We mostly follow chapters 8 and 24 of Jech, and occasionally the handbook chapter by Abraham and Magidor, or the survey *Notes on singular cardinal combinatorics* by Cummings.

1 Inductive formula

Some basic facts:

$\kappa^{\mathrm{cf}(\kappa)} > \kappa$ by König lemma. Also $\kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\kappa = 2^\kappa$.

The inductive formula for cardinal arithmetic reduces the value of a cardinal exponentiation κ^λ to values at smaller κ or λ. The cases when either κ or λ is finite are clear, so suppose they are infinite.

(i) If $\exists \eta < \kappa\ \eta^\lambda \geq \kappa$ (in particular if $\kappa \leq 2^\lambda$), then $\kappa^\lambda \leq (\eta^\lambda)^\lambda = \eta^\lambda$, so $\kappa^\lambda = \eta^\lambda$.

(ii) Now suppose $\forall \eta < \kappa\ \eta^\lambda < \kappa$. In particular $\lambda < 2^\lambda < \kappa$. Write $\kappa = \lim_{\alpha \to \mathrm{cf}(\kappa)} \kappa_\alpha$, where $\mathrm{cf}(\kappa) \leq \kappa$.

(a) If $\lambda < \mathrm{cf}(\kappa)$ then every $f : \lambda \to \kappa$ is bounded, so $\kappa^\lambda \leq \sum_{\eta<\kappa} \eta^\lambda = \kappa$, and $\kappa^\lambda = \kappa$. Note that this case happens if κ is regular.

(b) If $\lambda \geq \mathrm{cf}(\kappa)$, then to each $f : \lambda \to \kappa$ associate the sequence $(f_\alpha : \alpha < \mathrm{cf}(\kappa))$ where f_α is the truncation of f below κ_α, i.e., $\mathrm{dom}(f_\alpha) = \lambda$, $f_\alpha(i) = f(i)$ if $f(i) < \kappa_\alpha$ and $f_\alpha(i) = 0$ otherwise. The map $f \mapsto (f_\alpha)_{\alpha<\mathrm{cf}(\kappa)}$ is injective, and by assumption $\kappa_\alpha^\lambda < \kappa$, so $\kappa^\lambda \leq \kappa^{\mathrm{cf}(\kappa)}$. Therefore $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Next we show that 2^λ can also be reduced in some sense, in case λ is singular (if λ is regular then there isn't much to say due to Easton's theorem). Write $\lambda = \lim_{\alpha \to \mathrm{cf}(\lambda)} \lambda_\alpha$. To each $f : \lambda \to 2$, associate the sequence $(f_\alpha : \alpha < \mathrm{cf}(\lambda))$ where f_α is the restriction of f to λ_α. The map $f \mapsto (f_\alpha)_{\alpha<\mathrm{cf}(\lambda)}$ is injective, so $2^\lambda \leq \prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha}$; on the other hand $\prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha} \leq \prod_{\alpha<\mathrm{cf}(\lambda)} 2^\lambda = 2^\lambda$, so actually equality holds.

(i) If λ is a strong limit, i.e., $2^{\lambda_\alpha} < \lambda$ for all α, then $\prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha} \leq \prod_{\alpha<\mathrm{cf}(\lambda)} \lambda = \lambda^{\mathrm{cf}(\lambda)}$, so $2^\lambda = \lambda^{\mathrm{cf}(\lambda)}$.

(ii) If $2^{\lambda_\alpha} \geq \lambda$ for some α, then we actually have $2^{\lambda_\alpha} = 2^{\lambda_\alpha \cdot \mathrm{cf}(\lambda)} \geq \lambda^{\mathrm{cf}(\lambda)} > \lambda$ for large enough α. Consider the sequence $(2^{\lambda_\alpha} : \alpha < \mathrm{cf}(\lambda))$.

(a) If this sequence is eventually constant, i.e., there exists $\beta < \mathrm{cf}(\lambda)$ s.t. $2^{\lambda_\alpha} = 2^{\lambda_\beta}$ for all $\alpha \geq \beta$, then $\prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha} \leq \prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\beta} = 2^{\lambda_\beta \cdot \mathrm{cf}(\lambda)} = 2^{\lambda_\beta}$. Therefore $2^\lambda = 2^{\lambda_\beta}$.

(b) If this sequence is not eventually constant, then $\prod_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha} = (\sup_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha})^{\mathrm{cf}(\lambda)}$; $\leq$ is clear and $\geq$ follows by partitioning $\mathrm{cf}(\lambda)$ into $\mathrm{cf}(\lambda)$ many unbounded sets. Since the cardinal $\eta := \sup_{\alpha<\mathrm{cf}(\lambda)} 2^{\lambda_\alpha}$ has cofinality $\mathrm{cf}(\lambda)$, we can rewrite the equation as $2^\lambda = \eta^{\mathrm{cf}(\eta)}$.

A consequence is that any κ^λ is equal to the power of a regular cardinal or the value of gimel function at a singular cardinal.

2 Silver's theorem and Galvin-Hajnal theorem

There are two slightly different versions of Singular Cardinal Hypothesis.

$\mathsf{SCH1}$: if κ is a strong limit singular cardinal, then $2^\kappa = \kappa^+$.

$\mathsf{SCH2}$: if κ is a singular cardinal and $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

Both versions follow from GCH trivially. By the inductive formula, if κ is a strong limit then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$, so $\mathsf{SCH2}$ implies $\mathsf{SCH1}$. It is known that both $\neg\mathsf{SCH1}$ and $\neg\mathsf{SCH2}$ are equiconsistent with a measurable cardinal κ of Mitchell order κ^{++}; roughly speaking, κ has Mitchell order 0 if it is measurable, and Mitchell order 1 if the set of measurables below κ has measure 1, etc.

Silver's theorem says either form of SCH cannot fail for the first time at a singular cardinal of uncountable cofinality. The original proof used generic ultrapower; a purely combinatorial proof was found soon afterwards. For brevity we show the special case that $\mathsf{SCH1}$ cannot first fail at $\aleph_{\omega_1}$; the proof for $\mathsf{SCH2}$ and for general case is almost identical.

Thus we assume $\aleph_{\omega_1}$ is a strong limit and $\mathsf{SCH1}$ holds below $\aleph_{\omega_1}$, and derive that $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. By a standard closure argument, $C := \{\alpha < \omega_1 : \alpha \text{ is a limit and } \forall\beta < \alpha\ 2^{\aleph_\beta} < \aleph_\alpha\}$ is a club. Since $\mathsf{SCH1}$ holds below $\aleph_{\omega_1}$, we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha \in C$. It is thus sufficient to show that if $\{\alpha < \omega_1 : 2^{\aleph_\alpha} = \aleph_{\alpha+1}\}$ contains a club then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$; actually it suffices to assume this set is stationary.

To each $X \subseteq \aleph_{\omega_1}$ associate the sequence $f_X = \langle X \cap \aleph_\alpha : \alpha < \omega_1\rangle$, which belongs to $\prod_{\alpha<\omega_1} \mathcal{P}(\aleph_\alpha)$. Then $\mathcal{F} := \{f_X : X \subseteq \aleph_{\omega_1}\}$ is an *almost disjoint family*, i.e., for any different $f, g \in \mathcal{F}$, there exists $\beta < \omega_1$ s.t. $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \beta$. The map $X \mapsto f_X$ is clearly injective. Thus we are done if we can prove:

Lemma 2.1. *Suppose $\aleph_{\omega_1}$ is a strong limit, $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence of sets, $\mathcal{F} \subseteq \prod_{\alpha<\omega_1} A_\alpha$ is an almost disjoint family of functions, and $\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\}$ is stationary, then $|\mathcal{F}| \leq \aleph_{\omega_1+1}$.*

First we show the following related result.

Lemma 2.2. *Suppose $\aleph_{\omega_1}$ is a strong limit, $\langle A_\alpha : \alpha < \omega_1\rangle$ is a sequence of sets, $\mathcal{F} \subseteq \prod_{\alpha<\omega_1} A_\alpha$ is an almost disjoint family of functions, and $\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_\alpha\}$ is stationary, then $|\mathcal{F}| \leq \aleph_{\omega_1}$.*

Proof. Of course $S := \{\alpha < \omega_1 : \alpha \text{ is a limit and } |A_\alpha| \leq \aleph_\alpha\}$ is also stationary. We may assume every A_α is a cardinal and $A_\alpha \leq \aleph_\alpha$ for all $\alpha \in S$. For each $f \in \mathcal{F}$, since $f(\alpha) < \aleph_\alpha$ for all $\alpha \in S$, by Fodor's lemma there exists a stationary $T \subseteq S$ and $\beta < \omega_1$ s.t. $f(\alpha) < \aleph_\beta$ for all $\alpha \in T$. Associate to f the pair $(T, f{\restriction}T)$; note that $f{\restriction}T$ is a bounded function from T to $\aleph_{\omega_1}$. Since $\aleph_{\omega_1}$ is a strong limit, it is easy to count that the number of such pairs is $\aleph_{\omega_1}$. Finally, the map $f \mapsto (T, f{\restriction}T)$ is injective because if $f, g \in \mathcal{F}$ agree on a stationary set then they are the same. $\square$

Proof of Lemma 2.1. Again we may assume every A_α is a cardinal and the set $T := \{\alpha < \omega_1 : \alpha \text{ is a limit and } A_\alpha \leq \aleph_{\alpha+1}\}$ is stationary. We present two slightly different proofs.

(1) Let D be any ultrafilter on ω_1 that extends the club filter and contains T. Clearly if $f, g \in \mathcal{F}$ are different then they are still different modulo D, so we may regard $\mathcal{F}$ as a subset of the ultraproduct $\prod_\alpha A_\alpha/D$. Since the ultraproduct (and thus $\mathcal{F}$) is a linear order, it suffices to show

that for every $g \in \mathcal{F}$, the number of $f \in \mathcal{F}$ s.t. $f < g \mod D$ is at most $\aleph_{\omega_1}$: we can talk about cofinalities of arbitrary linear orders; if every proper initial segment of $\mathcal{F}$ has size at most $\aleph_{\omega_1}$ then the cofinality of $\mathcal{F}$ is at most $\aleph_{\omega_1+1}$, and then $\mathcal{F}$ has size at most $\aleph_{\omega_1+1}$.

Now if $f < g \mod D$ then $f(\alpha) < g(\alpha)$ on some stationary subset $S \subseteq T$. For $\alpha \in S$ we have $g(\alpha) < \aleph_{\alpha+1}$, and therefore $|g(\alpha)| \leq \aleph_\alpha$, so by Lemma 2.2 the set of such $f \in \mathcal{F}$ (for each fixed S) is at most $\aleph_{\omega_1}$. This finishes the proof of Lemma 2.1, hence Silver's theorem.

(2) For any $g \in \mathcal{F}$ and stationary $S \subseteq T$, define $\mathcal{F}_{g,S}$ as the set of all $f \in \mathcal{F}$ s.t. $f(\alpha) \leq g(\alpha)$ for all $\alpha \in S$, and let $\mathcal{F}_g = \bigcup \mathcal{F}_{g,S}$ where the union is taken over all stationary $S \subseteq T$. As before we have $|\mathcal{F}_{g,S}| \leq \aleph_{\omega_1}$ and thus $|\mathcal{F}_g| \leq \aleph_{\omega_1}$. Also notice that for any $f, g \in \mathcal{F}$, either $f \in \mathcal{F}_g$ or $g \in \mathcal{F}_f$.

Now inductively define a sequence $(f_\xi)_\xi$ as follows: if $\mathcal{F} \setminus \left(\bigcup_{\eta<\xi} \mathcal{F}_{f_\eta}\right) \neq \varnothing$, then let f_ξ belong to this set; this sequence is clearly injective. We claim that $f_{\aleph_{\omega_1+1}}$ cannot exist; otherwise $f_\xi \in \mathcal{F}_{f_{\aleph_{\omega_1+1}}}$ for all $\xi < \aleph_{\omega_1+1}$, and thus $|\mathcal{F}_{f_{\aleph_{\omega_1+1}}}| \geq \aleph_{\omega_1+1}$, a contradiction. Therefore, $\mathcal{F} = \bigcup_{\xi<\vartheta} F_{f_\xi}$ for some $\vartheta \leq \aleph_{\omega_1+1}$ and $|\mathcal{F}| \leq \aleph_{\omega_1+1}$. $\square$

Silver's Theorem can be vastly generalized; for example if $\{\alpha < \omega_1 : 2^{\aleph_\alpha} = \aleph_{\alpha+2}\}$ is stationary then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+2}$. The Galvin-Hajnal Theorem seems to be the ultimate result in this direction using elementary methods. It says if $\aleph_\alpha$ is a strong limit singular cardinal with uncountable cofinality, then $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$; note that this is nontrivial only if $\alpha < \aleph_\alpha$, namely α is not a fixed point of the aleph function.

Let us prove the special case that if $\aleph_{\omega_1}$ is a strong limit then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\omega_1})^+}$. Similar to Silver's Theorem, it suffices to show that if $\mathcal{F} \subseteq \prod_{\alpha<\omega_1} A_\alpha$ is an almost disjoint family where $|A_\alpha| < \aleph_{\omega_1}$ for all α then $|\mathcal{F}| < \aleph_{(2^{\omega_1})^+}$. There exists a function $\varphi : \omega_1 \to \omega_1$ s.t. $|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$. Consider the non-stationary ideal I_{NS} on ω_1. For functions $\varphi, \psi : \omega_1 \to \omega_1$, define $\varphi < \psi$ iff $\{\alpha < \omega_1 : \varphi(\alpha) \geq \psi(\alpha)\} \in I_{\mathrm{NS}}$; in other words $\varphi < \psi$ if ψ is bigger on a club set. Since clubs are closed under countable intersections, there cannot be infinite decreasing sequence; in other words the partial order $<$ is well-founded, so we can assign the *Galvin-Hajnal rank* $\|\varphi\|$ to each $\varphi : \omega_1 \to \omega_1$, defined recursively by $\|\varphi\| = \sup\{\|\psi\| + 1 : \psi < \varphi\}$. There are 2^{ω_1} many functions $\varphi : \omega_1 \to \omega_1$, so it's not hard to see that $\|\varphi\| < (2^{\omega_1})^+$ for any φ. It suffices to prove the following:

Lemma 2.3. *If $\varphi : \omega_1 \to \omega_1$ is a function and $\mathcal{F} \subseteq \prod_{\alpha<\omega_1} A_\alpha$ is an almost disjoint family s.t. $|A_\alpha| \leq \aleph_{\alpha+\varphi(\alpha)}$ for all α, then $|\mathcal{F}| \leq \aleph_{\omega_1+\|\varphi\|}$.*

This is proven by induction on the Galvin-Hajnal rank. Note that $\|\varphi\| = 0$ iff φ is zero on a stationary set, so the base case is exactly Lemma 2.2, which is where Fodor's lemma and hence the uncountable cofinality is used. The successor case is similar to the second proof of Lemma 2.1, but a bit more complicated and utilizes an auxiliary rank $\|\varphi\|_S$, which we now define. For each stationary set $S \subseteq \omega_1$, define the partial order $<_S$ on functions $\varphi : \omega_1 \to \omega_1$ by $\varphi <_S \psi$ iff $\{\alpha \in S : \varphi(\alpha) \geq \psi(\alpha)\} \in I_{\mathrm{NS}}$. Intuitively, we restrict the non-stationary ideal to S, and anything happening outside of S is ignored. The order $<_S$ is again well-founded, which gives rise to a rank $\|\varphi\|_S = \sup\{\|\psi\|_S + 1 : \psi <_S \varphi\}$. The basic Galvin-Hajnal rank is the case $S = \omega_1$. We list some properties of these ranks that will be useful. Both S and T range over stationary subsets of ω_1. We adopt the convention that 0 is not a limit ordinal.

1. If $S \subseteq T$ then $\varphi <_T \psi$ implies $\varphi <_S \psi$. Put another way, $<_S$ is finer than $<_T$. It can then be

proved by induction that $\|\varphi\|_T \leq \|\varphi\|_S$. Also $\varphi <_{S\cup T} \psi$ iff $\varphi <_S \psi$ and $\varphi <_T \psi$.

2. By induction on $\|\varphi\|_{S\cup T}$ we show that $\|\varphi\|_{S\cup T} = \min\{\|\varphi\|_S, \|\varphi\|_T\}$. Say $\|\varphi\|_S \leq \|\varphi\|_T$. To show $\|\varphi\|_{S\cup T} = \|\varphi\|_S$, consider any $\psi <_S \varphi$. There exists $\psi' <_T \varphi$ s.t. $\|\psi\|_S \leq \|\psi'\|_T$. Define ψ'' by $\psi'' = \psi$ on $S \setminus T$, $\psi'' = \psi'$ on $T \setminus S$ and $\psi'' = \max\{\psi, \psi'\}$ on $S \cap T$; the values outside of $S \cup T$ can be arbitrary. Then $\psi'' <_{S\cup T} \varphi$, $\|\psi''\|_S \geq \|\psi\|_S$ and $\|\psi''\|_T \geq \|\psi'\|_T \geq \|\psi\|_S$, so by induction hypothesis $\|\psi''\|_{S\cup T} \geq \|\psi\|_S$. Since this holds for arbitrary $\psi <_S \varphi$, we have $\|\varphi\|_{S\cup T} \geq \|\varphi\|_S$.

3. If $\{\alpha \in S : \varphi(\alpha) \text{ is not a limit ordinal}\} \in I_{\mathrm{NS}}$ (namely if $\varphi(\alpha)$ is a limit ordinal almost everywhere on S), then $\|\varphi\|_S$ is a limit ordinal, since if $\psi <_S \varphi$ then $\psi + 1 <_S \varphi$, where $\psi + 1$ means ψ plus 1 pointwise. Similarly, if $\{\alpha \in S : \varphi(\alpha) \text{ is not a successor ordinal}\} \in I_{\mathrm{NS}}$, then $\|\varphi\|_S = \|\varphi'\|_S + 1$, where φ' is the function defined by $\varphi(\alpha) = \varphi'(\alpha) + 1$ if $\varphi(\alpha)$ is a successor and arbitrarily elsewhere. In particular $\|\varphi\|_S$ is a successor ordinal.

4. Let $S = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a limit ordinal}\}$ and $T = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\}$. It follows from 2 and 3 that if $\|\varphi\|$ is a limit then $\|\varphi\| = \|\varphi\|_S$, since otherwise $\|\varphi\| = \|\varphi\|_T$ is a successor. Similarly, if $\|\varphi\|$ is a successor then $\|\varphi\| = \|\varphi\|_T$.

5. By induction, if φ is bounded by some $\beta < \omega_1$ on a stationary set then $\|\varphi\| \leq \beta$. Thus $\|\varphi\| = \beta$ iff $\varphi(\alpha) = \beta$ for stationarily many α and $\varphi(\alpha) > \beta$ almost everywhere else.

Proof of Lemma 2.3. WLOG each A_α is a cardinal at most $\aleph_{\alpha+\varphi(\alpha)}$. We prove the lemma by induction on $\|\varphi\|$. The base case $\|\varphi\| = 0$ is Lemma 2.2.

(i) $\|\varphi\|$ is a limit ordinal.

Then $S := \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a limit ordinal}\}$ is stationary. For every $f \in \mathcal{F}$ and $\alpha \in S$ we have $f(\alpha) < \aleph_{\alpha+\varphi(\alpha)}$, so there exists a ψ s.t. $\psi(\alpha) \leq \varphi(\alpha)$ for all $\alpha < \omega_1$, $\psi(\alpha) < \varphi(\alpha)$ for all $\alpha \in S$ (so $\psi <_S \varphi$), and $f \in \prod_{\alpha<\omega_1} \aleph_{\alpha+\psi(\alpha)}$. By property 4 we have $\|\varphi\| = \|\varphi\|_S$, so $\|\psi\| \leq \|\psi\|_S < \|\varphi\|_S = \|\varphi\|$. In particular $\|\psi\| < \|\varphi\|$, and we can apply induction hypothesis: $\mathcal{F} \cap \prod_{\alpha<\omega_1} \aleph_{\alpha+\psi(\alpha)}$ has size at most $\aleph_{\alpha+\|\psi\|} < \aleph_{\alpha+\|\varphi\|}$. Since there are just $2^{\omega_1} < \aleph_{\omega_1}$ many possibilities for ψ, we have $|\mathcal{F}| \leq \aleph_{\alpha+\|\varphi\|}$.

(ii) $\|\varphi\|$ is a successor ordinal.

Then $T = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\}$ is stationary and $\|\varphi\| = \|\varphi\|_T$. For each $g \in \mathcal{F}$ and stationary $S \subseteq T$ *such that* $\|\varphi\|_S = \|\varphi\|$, let $\mathcal{F}_{g,S} = \{f \in \mathcal{F} : \forall \alpha \in S\ f(\alpha) \leq g(\alpha)\}$ and $\mathcal{F}_g = \bigcup \mathcal{F}_{g,S}$ where the union is taken over all stationary $S \subseteq T$ satisfying $\|\varphi\|_S = \|\varphi\|$.

For each S as above, there is a function ψ s.t. $\psi(\alpha) \leq \varphi(\alpha)$ for all $\alpha < \omega_1$, $\psi(\alpha) < \varphi(\alpha)$ for all $\alpha \in S$ (so $\psi <_S \varphi$) and $|g(\alpha)| \leq \aleph_{\alpha+\psi(\alpha)}$ for all $\alpha < \omega_1$. Since $\|\psi\| \leq \|\psi\|_S < \|\varphi\|_S = \|\varphi\|$, the induction hypothesis gives $|\mathcal{F}_{g,S}| \leq \aleph_{\omega_1+\|\psi\|}$. Since there are only 2^{ω_1} many possibilities for ψ, and $\aleph_{\omega_1+\|\varphi\|}$ is a successor cardinal hence regular, we have $|\mathcal{F}_g| < \aleph_{\omega_1+\|\varphi\|}$.

Next we observe that for any $f, g \in \mathcal{F}$, if we let $S_1 = \{\alpha \in T : f(\alpha) \leq g(\alpha)\}$ and $S_2 = \{\alpha \in T : f(\alpha) \geq g(\alpha)\}$, then by property 2, either $\|\varphi\|_{S_1} = \|\varphi\|$ or $\|\varphi\|_{S_2} = \|\varphi\|$. Consequently, either $f \in \mathcal{F}_g$ or $g \in \mathcal{F}_f$.

Finally, inductively define a sequence $(f_\xi)_\xi$ as follows: if $\mathcal{F} \setminus \left(\bigcup_{\eta<\xi} \mathcal{F}_{f_\eta}\right) \neq \varnothing$, then let f_ξ belong to this set; this sequence is clearly injective. We claim that $f_{\aleph_{\omega_1+\|\varphi\|}}$ cannot exist; otherwise

$f_\xi \in \mathcal{F}_{f_{\aleph_{\omega_1+\|\varphi\|}}}$ for all $\xi < \aleph_{\omega_1+\|\varphi\|}$, and thus $|\mathcal{F}_{f_{\aleph_{\omega_1+\|\varphi\|}}}| \geq \aleph_{\omega_1+\|\varphi\|}$, contradicting the fact that $|\mathcal{F}_g| < \aleph_{\omega_1+\|\varphi\|}$ for all g. Therefore, $\mathcal{F} = \bigcup_{\xi<\vartheta} F_{f_\xi}$ for some $\vartheta \leq \aleph_{\omega_1+\|\varphi\|}$ and $|\mathcal{F}| \leq \aleph_{\omega_1+\|\varphi\|}$.

□

A natural question is whether there is an analogue of Galvin-Hajnal Theorem for singular cardinals of countable cofinality, say $\aleph_\omega$. This led Shelah to his celebrated pcf theory.

3 Basic notions of pcf theory

Shelah's pcf theory is the theory of *possible cofinalities* of ultraproducts: every linear order has a well-defined cofinality, and given a set A of regular cardinals, we consider the set $\mathrm{pcf}(A)$ of all possible cofinalities of the ultraproduct $\prod A/D$ for some ultrafilter D on A. It turns out this is deeply related to cardinal arithmetic. One of the most famous results in pcf theory is:

$(*)$ if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

This formula used to be displayed at the top of Shelah's website (before it was renovated). It is unknown whether the mysterious number 4 is optimal, although it is known that $2^{\aleph_\omega} = \aleph_{\alpha+1}$ is consistent relatively to large cardinals for any countable ordinal α.

We remark that what Shelah actually proved is the following ZFC result:

$\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max \mathrm{pcf}(A) < \aleph_{\omega_4}$,

where $A = \{\aleph_n : n \in \omega\}$, and $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ means the smallest cardinality of a cofinal subset of the partial order $([\aleph_\omega]^{\aleph_0}, \subseteq)$. It is easy to see that $2^{\aleph_0} < \aleph_\omega$ implies $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = |[\aleph_\omega]^{\aleph_0}| = \aleph_\omega^{\aleph_0}$, so this is a more refined version of $(*)$. The quantity $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ or equivalently $\max \mathrm{pcf}(A)$ should be viewed as more fundamental than $2^{\aleph_\omega}$ since it is much more robust; for example, it is not difficult to show that $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ cannot be changed by ccc forcing. However, for simplicity we shall follow Jech and just prove $(*)$.

We first introduce some basic notions. If A is any nonempty set, an ideal on A is a subset $I \subseteq \mathcal{P}(A)$ such that (i) $\varnothing \in I$, (ii) if $X, Y \in I$ then $X \cup Y \in I$, (iii) if $X \in I$ and $Y \subseteq X$ then $Y \in I$. If $A \notin I$ then we say I is a proper ideal. If I is an ideal then $\{A \setminus X : X \in I\}$ is a filter. We think of a set $X \in I$ as "small", "negligible" or "measure zero", while a set in the dual filter is "large" or "measure one"; I^+ denotes $\mathcal{P}(A) \setminus I$, and we call it the collection of I-positive sets. Any subset of $\mathcal{P}(A)$ generates a (possibly improper) ideal. If X is an I-positive set, we denote the (proper) ideal generated by $I \cup \{A \setminus X\}$ as $I \upharpoonright X$, for which X is still a positive set.

If I is an ideal on A and f, g are ordinal functions on A, i.e., $\mathrm{dom}(f) = \mathrm{dom}(g) = A$ and $f(a), g(a) \in \mathrm{Ord}$ for all $a \in A$, then we define

$f <_I g \Leftrightarrow \{a \in A : f(a) \geq g(a)\} \in I$

$f \leq_I g \Leftrightarrow \{a \in A : f(a) > g(a)\} \in I$

$f =_I g \Leftrightarrow \{a \in A : f(a) \neq g(a)\} \in I$

$<_I$ is a strict partial order as long as I is proper, $\leq_I$ is a pre-order that is finer than $<_I$, and $=_I$ is an equivalence relation. There are some simple relations between these, for example if $f <_I g$

and $g \leq_I h$ then $f <_I h$. Be careful that $f <_I g$ is not the same as $f \leq_I g \wedge \neg(f =_I g)$; say I is the ideal consisting of all finite subsets of A, then $f <_I g$ means $f(a) < g(a)$ on a cofinite set, while $f \leq_I g \wedge \neg(f =_I g)$ means $f(a) < g(a)$ on an infinite set. If F is a filter on A then $<_F$ means $<_I$ where I is the dual ideal. We are most interested in the case $A = \{\aleph_n : n \in \omega\}$ and I some proper ideal that at least contains all the finite sets.

By a *directed relation*, we mean a binary relation $(\mathbb{P}, E)$ that satisfies:

(i) It is transitive, namely $pEq \wedge qEr \to pEr$.

(ii) For any $p, q \in \mathbb{P}$ there is $r \in \mathbb{P}$ s.t. pEr and qEr.

(iii) There is no maximal element, that is a p such that qEp for all $q \in \mathbb{P}$. It follows that every element has a proper successor, meaning $\forall p \in \mathbb{P} \exists q \in \mathbb{P}\ pEq \wedge \neg(qEp)$, because there exists q such that $\neg qEp$, and by (ii) there exists a q' such that qEq' and pEq', and thus $\neg q'Ep$.

We don't assume anything about reflexivity, in order to treat pre-order and strict partial order simultaneously. A subset $X \subseteq \mathbb{P}$ is *unbounded* if $\forall p \in \mathbb{P} \exists q \in X \neg(qEp)$, and is *cofinal* or *dominating* if $\forall p \in \mathbb{P} \exists q \in X (pEq)$. A cofinal set is unbounded because of (ii) above. We define the bounding number $\mathfrak{b}(\mathbb{P}, E) = \min\{|X| : X \subseteq \mathbb{P} \text{ is unbounded}\}$ and the dominating number $\mathfrak{d}(\mathbb{P}, E) = \min\{|X| : X \subseteq \mathbb{P} \text{ is dominating}\}$; the latter is more often denoted $\mathrm{cf}(\mathbb{P}, E)$. When applied to $(\omega^\omega, <^*)$, where $<^*$ denotes eventual dominance, this gives the classical cardinal characteristics $\mathfrak{b}$ and $\mathfrak{d}$ on continuum. Call a subset $X \subseteq \mathbb{P}$ linear if the induced relation (X, E) satisfies that for any different $p, q \in X$, exactly one of pEq and qEp holds. Below are some simple observations; all relations are assumed to be directed.

1. Always $\mathfrak{b}(\mathbb{P}, E) \leq \mathfrak{d}(\mathbb{P}, E)$. Equality holds iff $(\mathbb{P}, E)$ has a linear cofinal subset; in this case we sometimes denote both cardinal as $\mathrm{tcf}(\mathbb{P}, E)$, the "true cofinality". An example of a directed poset without true cofinality is $\omega \times \omega_1$ with the product order.

2. If X is a cofinal subset of $\mathbb{P}$, then $\mathfrak{d}(\mathbb{P}, E) = \mathfrak{d}(X, E)$. If $(\mathbb{P}, E')$ is finer than $(\mathbb{P}, E)$, namely $pEq \to pE'q$, then $\mathfrak{b}(\mathbb{P}, E) \leq \mathfrak{b}(\mathbb{P}, E') \leq \mathfrak{d}(\mathbb{P}, E') \leq \mathfrak{d}(\mathbb{P}, E)$. These are special cases of *Tukey reduction*. In particular, if $(\mathbb{P}, E)$ has true cofinality then all four cardinals are equal, so $\mathrm{cf}(\mathbb{P}, E) = \mathrm{cf}(\mathbb{P}, E')$ for any directed relation E' that is finer than E; we will use this all the time.

3. Suppose I is a proper ideal on A and F is a set of ordinal functions that is directed in $<_I$, namely for any $f, g \in F$ there exists $h \in F$ such that $f <_I h$ and $g <_I h$; in particular for any $f \in F$ there exists $g \in F$ such that $f <_I g$. Then it's not difficult to check that a set $X \subseteq F$ is unbounded/cofinal in $(F, <_I)$ iff it is unbounded/cofinal in $(F, \leq_I)$, so their bounding/dominating numbers are the same. In practice F is usually a product of limit ordinals or a $<_I$-increasing sequence, and hence directed, so there is no ambiguity when we say something like "I is λ-directed."

4. The bounding number is always regular, while the dominating number may not be, but it is regular if $\mathbb{P}$ is a linear order, or more generally when it has true cofinality. Then it can be shown that $(\mathbb{P}, E)$ has a well-ordered cofinal subset, say by defining a cofinal sequence inductively, and that $\mathrm{cf}(\mathbb{P}, E)$ is equal to the cofinality of that well-order, which must be regular.

For a cardinal λ, we say that $(\mathbb{P}, E)$ is λ-*directed* if it is directed and $\mathfrak{b}(\mathbb{P}, E) \geq \lambda$; by definition, this means that for any $X \subseteq \mathbb{P}$ with $|X| < \lambda$, there exists $p \in \mathbb{P}$ such that qEp for all $q \in X$. If λ is singular and $(\mathbb{P}, E)$ is λ-directed, then it is actually λ^+-directed.

Now we can define the central notion in pcf theory. Let A be a nonempty set of infinite regular cardinals. Denote by $\prod_{a \in A} a$ or $\prod A$ the set of all functions such that $\mathrm{dom}(f) = A$ and $\forall a \in A\ f(a) \in a$; be careful that if $A = \{\aleph_n : n \in \omega\}$ then f is a function on A rather than ω, although we can certainly translate between them. Let $\prod_{a \in A} a/D$ or $\prod A/D$ be the ultraproduct by D, consisting of equivalence classes of functions; since each $a \in A$ is a regular cardinal, in particular a linear order, so is the ultraproduct, and we can talk about its cofinality. It is not hard to see that $\mathrm{cf}(\prod_{a \in A} a/D) = \mathrm{cf}(\prod_{a \in A} a, <_D)$, and we use them interchangeably.

Definition 3.1. $\mathrm{pcf}(A) := \{\mathrm{cf}(\prod_{a \in A} a/D) : D \text{ is an ultrafilter on } A\}$.

We list several properties of the pcf operation, some of them to be proven later.

1. $\mathrm{pcf}(A) \supseteq A$ because we allow D to be principal, and if $\lambda \in \mathrm{pcf}(A)$ then $\lambda \geq \min A$, because if $\kappa < \min A$ then the pointwise supremum of κ many functions in $\prod A$ is again in $\prod A$, so in any ultraproduct, any κ many functions have an upper bound.

2. If $A \subseteq B$ then $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(B)$. This is because there is a natural bijection between ultrafilters on A and ultrafilters on B that are concentrated on (i.e. contain) A.

3. $\mathrm{pcf}(A \cup B) = \mathrm{pcf}(A) \cup \mathrm{pcf}(B)$. This is because an ultrafilter on $A \cup B$ is either concentrated on A or on B.

4. If $|\mathrm{pcf}(A)| < \min A$ then $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$. This will be proven later, and is basically because an ultraproduct of ultraproducts is again an ultraproduct. Note that a trivial bound for $|\mathrm{pcf}(A)|$ is $2^{2^{|A|}}$. In our main application A will be $\{\aleph_n : n \in \omega\}$, and will satisfy $2^{2^{|A|}} < \min A$ provided $\aleph_\omega$ is a strong limit and we delete the first several points from A, which has minimal effect on $\mathrm{pcf}(A)$.

5. Call a set A of regular cardinals an *interval* if whenever $a, b \in A$ and c is some regular cardinal s.t. $a < c < b$, then $c \in A$. If A is an interval of regular cardinals and $2^{|A|} < \min A$, then $\mathrm{pcf}(A)$ is also an interval.

A set A of regular cardinals is called *progressive* if $|A| < \min A$. For example, if $\aleph_\alpha$ is a singular cardinal and $\alpha < \aleph_\alpha$ (i.e., $\aleph_\alpha$ is not an alpeh fixed point) then $[|\alpha|^+, \aleph_\alpha) \cap \mathrm{Reg}$ is progressive, where Reg is the class of regular cardinals. This is the standard assumption used in many texts on pcf theory, but it makes proofs harder, for example we only get a weak version of property 4. Thus we follow Jech and constantly assume $2^{|A|} < \min A$ or even stronger conditions.

Properties 1-4 can be interpreted topologically; this is not needed in the proofs but may provide some intuition. First notice that if $|\mathrm{pcf}(A)| < \min A$ then for any $X \subseteq \mathrm{pcf}(A)$, we have $\mathrm{pcf}(X) \subseteq \mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$, and moreover $|\mathrm{pcf}(X)| \leq |\mathrm{pcf}(A)| < \min A \leq \min X$, so $\mathrm{pcf}(\mathrm{pcf}(X)) = \mathrm{pcf}(X)$. If we artificially let $\mathrm{pcf}(\varnothing) = \varnothing$, then pcf is an operation on the power set of $\overline{A} := \mathrm{pcf}(A)$. The above properties tells us this is a *closure operator*, namely there is a topology on $\overline{A}$ whose closed sets are exactly those X s.t. $\mathrm{pcf}(X) = X$, and the pcf operation is exactly topological closure.

The significance of 5 is that if $\mathrm{pcf}(A)$ is an interval, then a bound on $|\mathrm{pcf}(A)|$ gives a bound on $\sup \mathrm{pcf}(A)$. We can now outline the first application of pcf theory: if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$; this is exactly the analogue of Galvin-Hajnal for countable cofinality. The proof will proceed as follows. Let $A = \{\aleph_n : n \in \omega\}$; a few facts not mentioned above are that we have a nontrivial bound $|\mathrm{pcf}(A)| \leq 2^{|A|}$, $\mathrm{pcf}(A)$ actually has a maximal element, and that this element is equal to $2^{\aleph_\omega}$. Since $\mathrm{pcf}(A)$ is an interval starting at $\aleph_0$ and has at most $2^{\aleph_0}$ elements, if its maximal element is $\aleph_\alpha$ then we must have $\alpha < (2^{\aleph_0})^+$.

A better bound on $\mathrm{pcf}(A)$ will clearly give us a better bound on $2^{\aleph_\omega}$. Using the existence of so-called transitive generators, a pretty technical theorem even within pcf theory, we will show that $|\mathrm{pcf}(A)| < \omega_4$, and thus $2^{\aleph_\omega} < \aleph_{\omega_4}$. This is only tighter than the first bound if $2^{\aleph_0} \geq \omega_4$, which may not seem super impressive, but keep in mind that using the full strength and technicality of pcf theory one can prove $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max \mathrm{pcf}(A) < \aleph_{\omega_4}$, which gives a reasonable bound even in case $2^{\aleph_0}$ is weakly inaccessible.

4 Ultraproducts, exact upper bounds

We first prove property 4, that if $|\mathrm{pcf}(A)| < \min A$ then $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$. It essentially follows from general facts about ultraproducts of structures.

Proposition 4.1. *Suppose we are given the following data: $(M_i : i \in I)$ are structures in the same language; D_j is an ultrafilter on I for each $j \in J$; denote the ultraproduct $\prod_i M_i/D_j$ by N_j; finally E is an ultrafilter on J. Then there exists an ultrafilter U on $I \times J$ such that $\prod_j N_j/E \simeq \prod_{i,j} M_i/U$.*

$\prod_{i,j} M_i$ is the set of functions f on $I \times J$ s.t. $f(i,j) \in M_i$. So an iterated ultraproduct can actually be rewritten as a single ultraproduct of the same set of models. A variant of this proposition is used in the study of measurable cardinals.

Proof. For any $Z \subseteq I \times J$ and $j \in J$, define $Z_j = \{i \in I : (i,j) \in Z\}$. Let

$$U = \{Z \subseteq I \times J : \{j \in J : Z_j \in D_j\} \in E\}.$$

It is not hard to verify that U is an ultrafilter. If we picture $I \times J$ as a rectangle (J being the length) then a set Z is large iff E-almost all vertical sections are large w.r.t. the measure D_j on that section.

For any $f \in \prod_{i,j} M_i$, define $f_j(i) = f(i,j)$, so $f_j \in \prod_i M_i$ and $[f_j]_{D_j} \in N_j$. Consider the map $\pi : \prod_{i,j} M_i \to \prod_j N_j,\ f \mapsto ([f_j]_{D_j})_j$. The definition of U readily implies that $f = g \mod U$ iff $\pi(f) = \pi(g) \mod E$, and thus π induces an injection from $\prod_{i,j} M_i/U$ to $\prod_j N_j/E$, denoted $\bar{\pi}$. Since π is easily seen to be surjective, so is $\bar{\pi}$. Elementarity is similar, for example $\prod_{i,j} M_i/U \models \varphi([f]_U)$ iff $\{(i,j) \in I \times J : M_i \models \varphi(f(i,j))\} \in U$ iff $\{j \in J : \{i \in I : M_i \models \varphi(f_j(i))\} \in D_j\} \in E$ iff $\{j \in J : N_j \models \varphi([f_j]_{D_j})\} \in E$ iff $\prod_j N_j/E \models \varphi(\pi(f))$. □

If $p : I \to J$ is a map and D is an ultrafilter on I, we define its push-forward to be $p_*(D) = \{Y \subseteq J : p^{-1}(Y) \in D\}$, which is an ultrafilter, and is non-principal if the preimage of every $j \in J$ is D-small. Push-forward actually makes sense for arbitrary filters, or indeed any "type 2" objects such as measures, functionals, etc.

Proposition 4.2. *Suppose $p : I \to J$ is a map, D is an ultrafilter on I, and $(M_j : j \in J)$ are structures in the same language. Then $\prod_j M_j/p_*(D)$ elementarily embeds into $\prod_i M_{p(i)}/D$.*

Proof. The map $\tau : \prod_j M_j \to \prod_i M_{p(i)},\ f \mapsto f \circ p$ induces an elementary embedding. For example $\prod_j M_j/p_*(D) \models \varphi([f])$ iff $\{j \in J : M_j \models \varphi(f(j))\} \in p_*(D)$ iff $\{i \in I : M_{p(i)} \models \varphi(f(p(i)))\} \in D$ iff $\prod_i M_{p(i)}/D \models \varphi([f \circ p])$. □

Corollary 4.3. *If A is a set of regular cardinals and $|\mathrm{pcf}(A)| < \min A$, then $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.*

Proof. Let $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A))$, so there exist an ultrafilter E on $\mathrm{pcf}(A)$ such that $\lambda = \mathrm{cf}(\prod_{\nu \in \mathrm{pcf}(A)} \nu/E)$. For every $\nu \in \mathrm{pcf}(A)$ there is an ultrafilter D_ν on A such that $\nu = \mathrm{cf}(\prod_{a \in A} a/D_\nu)$. For brevity, any subscript a ranges over $a \in A$ and ν ranges over $\nu \in \mathrm{pcf}(A)$.

Denote $N_\nu = \prod_a a/D_\nu$. Since $\mathrm{cf}(N_\nu) = \nu$, it is not hard to see that $\prod_\nu \nu/E$ cofinally embeds into $\prod_\nu N_\nu/E$, and thus $\lambda = \mathrm{cf}(\prod_\nu N_\nu/E)$. By Proposition 4.1 there is an ultrafilter U on $A \times \mathrm{pcf}(A)$ such that $\prod_{a,\nu} a/U \simeq \prod_\nu N_\nu/E$, so $\lambda = \mathrm{cf}(\prod_{a,\nu} a/U)$.

Consider the map $p : A \times \mathrm{pcf}(A) \to A$ and let $V = p_*(U)$. By Proposition 4.2 the map $\tau : \prod_a a \to \prod_{a,\nu} a, f \mapsto f \circ p$ induces an elementary embedding $\bar{\tau} : \prod_a a/V \to \prod_{a,\nu} a/U$. We claim that this embedding is cofinal. If $f \in \prod_a a$, then $\tau(f)$ is simply $\mathrm{pcf}(A)$ many copies of f. If $g \in \prod_{a,\nu} a$, then since $|\mathrm{pcf}(A)| < \min A$ and A consists of regular cardinals, if we define $f(a) = \sup_\nu g(a,\nu)$ then $f \in \prod_a a$; clearly $\tau(f) \geq g$. Thus $\bar{\tau}$ is cofinal, and $\lambda = \mathrm{cf}(\prod_a a/V)$. □

Our next goal is to show property 5, that the pcf of an interval is an interval. For that we need to first study exact upper bounds of ordinal functions. We will show that, under the simplifying assumption that $2^{|A|} < \min A$, eub almost always exists. This has another important corollary: the existence of pcf generators, which gives a nontrivial bound $|\mathrm{pcf}(A)| \leq 2^{|A|}$. Then once we connect $\max \mathrm{pcf}(A)$ to $2^{\aleph_\omega}$, we will be able to give the first application of pcf theory to cardinal arithmetic.

Suppose I is a proper ideal on A and F is a set of ordinal functions on A; as mentioned before F is almost always directed in $<_I$. We call $g \in \mathrm{Ord}^A$ an *upper bound* of F (w.r.t. I) if $\forall f \in F\ f \leq_I g$. Again, for directed F it doesn't matter whether we use $\leq_I$ or $<_I$ in the definition. We call g a *least upper bound* if it is an upper bound, and if h is also an upper bound then $g \leq_I h$; this is the same as saying that g is a minimal upper bound, namely if h is an upper bound and $h \leq_I g$ then $h =_I g$, because $(\mathrm{Ord}^A, \leq_I)$ is a lattice (for general poset lub and mub are different). Lub is clearly unique up to $=_I$ if it exists. We call g an *exact upper bound* of F if F is cofinal in $(\prod_a g(a), <_I)$, or equivalently (for directed F) if $h <_I g$ then there exists $f \in F$ s.t. $h <_I f$. For example, if $g(a)$ is a limit ordinal for every a then g is an eub of $F = \prod_a g(a)$. Also if F is directed in $<_I$ and g is an eub, then $g(a)$ is a limit ordinal for I-almost all a, so WLOG we may assume $g(a)$ is a limit everywhere.

Let $A = \omega$, I be the ideal of finite sets and f_n be the constant function on ω with value n. Then $\langle f_n : n < \omega \rangle$ doesn't even have a lub, as is easily checked (we are confusing $\langle f_n : n < \omega \rangle$ with $\{f_n : n < \omega\}$ as in standard practice). However, it turns out any $<_I$-increasing sequence that is sufficiently long does have an eub. The proof uses elementary sub-models, as is common in pcf theory. The size of the model is often chosen to be something between $2^{|A|}$ and $\min A$.

Theorem 4.4. *Suppose A is a set, I is a proper ideal, $\lambda > 2^{|A|}$ is regular and $F = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of ordinal functions on A, namely if $\xi < \eta$ then $f_\xi <_I f_\eta$. Then F has an exact upper bound.*

Proof. Let M be an elementary sub-model of H_θ for some large enough θ, such that $|M| = 2^{|A|}$, ${}^{|A|}M \subseteq M$, and $I, F \in M$; it follows that $A \subseteq M$. For each $\xi < \lambda$ define an ordinal function g_ξ on A by

$g_\xi(a) =$ the least $\beta \in M$ s.t. $\beta \geq f_\xi(a)$.

First note that such a β exists (even though f_ξ may not be in M), because $\sup_{\xi<\lambda} f_\xi(a)$ is definable for each a and thus in M by elementarity. We have $g_\xi \in {}^{|A|}M$, and hence $g_\xi \in M$ since M is closed under sequences of length $|A|$. Recall $|M| = 2^{|A|} < \lambda$, so there exists a $g \in M$ s.t. $g_\xi = g$ for unboundedly many ξ. Since $g \geq f_\xi$ pointwise for unboundedly many ξ and $F = \langle f_\xi : \xi < \lambda\rangle$ is $<_I$-increasing, g is an upper bound of F.

Next we show g is a lub; by elementarity it is enough to show this in M. Suppose $h \in M$ and h is an upper bound of F; choose any ξ s.t. $g = g_\xi$; then by definition $g_\xi \leq_I h$, so $g \leq_I h$.

Finally we show that under the assumptions in the theorem, an lub g is automatically an eub. Suppose otherwise, then there exists $h <_I g$ s.t. $h \not<_I f_\xi$ for any $\xi < \lambda$, so there exists $X_\xi \in I^+$ s.t. $f_\xi(a) \leq h(a)$ for $a \in X_\xi$. Since $\lambda > 2^{|A|}$ and is regular, there exists $X \in I^+$ s.t. $X = X_\xi$ for an unbounded set Z of ξ, so $f_\xi(a) \leq h(a)$ for all $a \in X$ and $\xi \in Z$. It follows that if we define g' by $g'(a) = h(a)$ for $a \in X$ and $g'(a) = g(a)$ elsewhere, then g' is an upper bound of $\langle f_\xi : \xi \in Z\rangle$, and hence of F, contradicting that g is an lub. $\square$

Let's show the usefulness of eub by proving a weak version of property 5 of pcf operation, which suffices for our application to $\aleph_\omega$.

Corollary 4.5. *If A is an interval of regular cardinals and $(2^{|A|})^+ = \min A$, then $\mathrm{pcf}(A)$ is an interval.*

Proof. Suppose $\mu \in \mathrm{pcf}(A)$ and $\min A \leq \lambda < \mu$ is regular; we shall show that $\lambda \in \mathrm{pcf}(A)$. Let D be an ultrafilter on A and $\langle f_\xi : \xi < \mu\rangle$ be $<_D$-increasing and cofinal in $\prod A$. By the theorem $\langle f_\xi : \xi < \lambda\rangle$ has an eub g, which we may assume satisfies $g(a) < a$, because f_λ is an upper bound and $f_\lambda \in \prod A$. Therefore, $\mathrm{cf}(\prod_{a\in A} g(a), <_D) = \lambda$, which is the same as saying $\mathrm{cf}(\prod_{a\in A} g(a)/D) = \lambda$; clearly we also have $\mathrm{cf}(\prod_{a\in A} \mathrm{cf}(g(a))/D) = \lambda$. Note that $\{a \in A : \mathrm{cf}(g(a)) \geq (2^{|A|})^+\} \in D$, since otherwise $\prod_{a\in A} \mathrm{cf}(g(a))/D$ has at most $(2^{|A|})^{|A|} = 2^{|A|}$ many elements. Since also $\mathrm{cf}(g(a)) < a$ and A is an interval, we have $\{a \in A : \mathrm{cf}(g(a)) \in A\} \in D$.

Consider the map $p : A \to A$ that sends a to $\mathrm{cf}(g(a))$ in case $\mathrm{cf}(g(a)) \in A$, and to arbitrary value otherwise. Consider the embedding $\prod_{a\in A} a/p_*(D) \to \prod_{a\in A} p(a)/D$ from Proposition 4.2; it is induced by $f \mapsto f \circ p$. We claim that this embedding is cofinal, and thus $\mathrm{cf}(\prod_{a\in A} a/p_*(D)) = \mathrm{cf}(\prod_{a\in A} p(a)/D) = \mathrm{cf}(\prod_{a\in A} \mathrm{cf}(g(a))/D) = \lambda$. For $f \in \prod_{a\in A} p(a)$ consider f' defined by $f'(a) = \sup\{f(b) : p(b) = a\} < a$ for each a; because $|\{f(b) : p(b) = a\}| \leq |A| < \min A$ and each $a \in A$ is regular, we have $f'(a) \in a$, in other words $f' \in \prod_{a\in A} a$, and $f \leq f' \circ p$ pointwise by definition. $\square$

Consequently, if $2^{\aleph_0} < \aleph_\omega$ then $\mathrm{pcf}\{\aleph_n : n < \omega\}$ is an interval: if we let $\aleph_k = 2^{\aleph_0}$ then $\mathrm{pcf}\{\aleph_n : k+1 \leq n < \omega\}$ is an interval, which easily implies $\mathrm{pcf}\{\aleph_n : n < \omega\}$ is an interval. But to show that, e.g., $\mathrm{pcf}\{\aleph_{\omega+n} : 1 \leq n < \omega\}$ is an interval using the above result, we need the ad hoc assumption that $2^{\aleph_0} = \aleph_{\omega+k}$ for some k. Later we will see that actually $2^{|A|} < \min A$ suffices; the key is to show that if $\langle f_\xi : \xi < \mu\rangle$ has many *good points*, then its eub g satisfies

$\mathrm{cf}(g(a)) \geq \min A$ for most a. Note that we cannot expect any $\langle f_\xi : \xi < \mu \rangle$ to work. For instance, suppose $A = \{\aleph_{\omega+n} : 1 \leq n < \omega\}$ and $B = \{\aleph_n : 1 \leq n < \omega\}$. Any sequence $\langle f_\xi : \xi < \mu \rangle \subseteq \prod B$ can be regarded as a sequence in $\prod A$, but its eub g cannot satisfy $\mathrm{cf}(g(a)) \geq \min A = \aleph_{\omega+1}$.

5 Pcf generators

If $A = \{\aleph_n : n \in \omega\}$, then by the previous section we know that $\mathrm{pcf}(A)$ is an interval, which means it must contain $\aleph_{\omega+1}$. Because for any non-principal ultrafilter D on A, the partial order $(\prod A, <_D)$ is $\aleph_{\omega+1}$-directed. Indeed, suppose $X \subseteq \prod A$ and $|X| = \aleph_k$ for some k, then we can define an upper bound g by $g(\aleph_n) = 0$ for $n \leq k$ and $g(\aleph_n) = \sup\{f(\aleph_n) : f \in X\}$ for $n > k$. Thus $\mathrm{cf}(\prod A, <_D) \geq \aleph_k$ for every k, which means $\mathrm{cf}(\prod A, <_D) \geq \aleph_{\omega+1}$.

It turns out we can do better: there is a set $B \subseteq A$ such that $(\prod B, <_I)$ already has true cofinality $\aleph_{\omega+1}$, where I is the ideal of finite subsets of B; it follows that $\mathrm{cf}(\prod A, <_D) = \aleph_{\omega+1}$ for any non-principal ultrafilter D on A that contains B. In fact there is such a B that is maximal modulo finite sets, and we call this canonical object the pcf generator at $\aleph_{\omega+1}$.

This suggests that we should study not only $(\prod A, <_D)$ for ultrafilters D, but also $(\prod A, <_I)$ for general ideals, for example when does it have true cofinality. Recall that if I is a proper ideal on A and $X \in I^+$, then $I \restriction X$ is the (proper) ideal generated by $I \cup \{A \setminus X\}$. Also if $F \subseteq \prod A$ is cofinal in $<_I$, then it remains so in $<_{I \restriction X}$, or indeed in any partial order finer than $<_I$, such as $<_D$ where D is an ultrafilter whose dual ideal contains I. Similarly, if F is bounded in $<_I$ then the same holds for $<_{I \restriction X}$. It is not hard to see that $(\prod A, <_{I \restriction X})$ and $(\prod X, <_{I_X})$ are essentially the same, where $I_X = I \cap \mathcal{P}(X)$; in particular they have the same bounding and dominating number. We prefer to write $(\prod A, <_{I \restriction X})$ so as to stick with ideals on A and avoid ambiguity.

Let I be the ideal of finite sets on $A = \{\aleph_n : n \in \omega\}$. As mentioned above, there exists a $B \subseteq A$ that is maximal modulo I such that $\mathrm{tcf}(B, <_{I_B}) = \aleph_{\omega+1}$, or equivalently $\mathrm{tcf}(A, <_{I \restriction B}) = \aleph_{\omega+1}$. There are two cases. First, B could be (modulo I) all of A, in which case $I \restriction B = I$ and $\aleph_{\omega+1} = \max \mathrm{pcf}(A)$. If B is not everything, it turns out if we consider the ideal J generated by $I \cup \{B\}$, then $(A, <_J)$ is $\aleph_{\omega+2}$-directed, and that there exists a $C \subseteq A$ such that $\mathrm{tcf}(A, <_{J \restriction C}) = \aleph_{\omega+2}$, and moreover there is such a C maximal modulo J. This analysis can be continued until the point where we reach the maximal element of $\mathrm{pcf}(A)$, and it gives us a fairly good picture of what $(A, <_I)$ for an arbitrary ideal I might look like.

In the rest of this section, let A be a set of infinite regular cardinals such that $2^{|A|} < \min A$; we don't need A or $\mathrm{pcf}(A)$ to be an interval.

Lemma 5.1 (Scale trichotomy)**.** *If $F = \langle f_\xi : \xi < \lambda \rangle \subseteq \prod A$ is $<_I$-increasing and $\lambda > 2^{|A|}$, then exactly one of the following holds: (i) F is cofinal in $(\prod A, <_I)$; (ii) F is bounded in $(\prod A, <_I)$; (iii) there exist disjoint $X, Y \in I^+$ s.t. $X \cup Y = A$, F is cofinal in $(\prod A, <_{I \restriction X})$ and F is bounded in $(\prod A, <_{I \restriction Y})$.*

Proof. F has an eub g, and we may assume $g(a) \leq a$ for all $a \in A$. Let $X = \{a \in A : g(a) = a\}$ and $Y = \{a \in A : g(a) < a\}$. Since F is cofinal in $(\prod_{a \in A} g(a), <_I)$, it is cofinal in $(\prod A, <_{I \restriction X})$; on the other hand g' is an upper bound of F in $(\prod A, <_{I \restriction Y})$ where $g'(a) = g(a)$ for $a \in Y$ and $g(a) = 0$ elsewhere. $\square$

We say that I has a λ-*scale* if $\mathrm{tcf}(\prod A, <_I) = \lambda$, namely there exists a sequence $\langle f_\xi : \xi < \lambda\rangle \subseteq \prod A$ that is $<_I$-increasing and cofinal in $\prod A$. We say I has a λ-scale on $X \in I^+$ if $I \restriction X$ has a λ-scale.

A proper ideal I is called λ-directed if $(\prod A, <_I)$ is λ-directed, which by definition means $\mathfrak{b}(\prod A, <_I) \geq \lambda$, namely any $X \subseteq \prod A$ s.t. $|X| < \lambda$ has an upper bound in $(\prod A, <_I)$. For singular λ, if I is λ-directed then it is λ^+-directed.

Lemma 5.2 (Ideal trichotomy)**.** *If $\lambda > 2^{|A|}$ is regular and the proper ideal I is λ-directed, then exactly one of the following holds: (i) I has a λ-scale; (ii) I is λ^+-directed; (iii) there exist disjoint $X, Y \in I^+$ s.t. $X \cup Y = A$, $I \restriction X$ has a λ-scale and $I \restriction Y$ is λ^+-directed.*

Proof. Suppose I doesn't have a λ-scale and isn't λ^+-directed, so there is an unbounded subset of $(\prod A, <_I)$ of cardinality λ. Using the fact that I is λ-directed, we can construct a sequence $F = \langle f_\xi : \xi < \lambda\rangle \subseteq \prod A$ that is increasing in $<_I$ and unbounded in $(\prod A, <_I)$; it cannot be cofinal since I doesn't have λ-scale. By the previous lemma there exist disjoint $X, Y \in I^+$ s.t. F is cofinal in $(\prod A, <_{I \restriction X})$, in other words I has a λ-scale on X, and also F is bounded in $(\prod A, <_{I \restriction Y})$. The fact that a particular F is bounded doesn't mean $I \restriction Y$ must be λ^+-directed, so we need to work a bit harder.

Let $\mathcal{Z}$ be the collection of all $Z \in I^+$ for which I has a λ-scale on Z, and for each such Z let F_Z be a $<_{I \restriction Z}$-increasing and cofinal sequence. Since $|\mathcal{Z}| \leq |\mathcal{P}(A)| < \lambda$ and $|F_Z| = \lambda$ for each Z, we have $|\bigcup_{Z \in \mathcal{Z}} F_Z| = \lambda$. Since I is λ-directed, we can construct an $<_I$-increasing sequence $F = \langle f_\xi : \xi < \lambda\rangle \subseteq \prod A$ s.t. for any $Z \in \mathcal{Z}$ and any $f \in F_Z$, there exists ξ s.t. $f <_I f_\xi$. By construction F is a scale in $(\prod A, <_{I \restriction Z})$ for every $Z \in \mathcal{Z}$; in particular F is unbounded in $<_I$ (otherwise it would be bounded in $<_{I \restriction Z}$). Let $X, Y \in I^+$ be as in the previous paragraph. We claim that $I \restriction Y$ is λ^+-directed. Otherwise, we repeat the argument to find $Z \in (I \restriction Y)^+$ s.t. $I \restriction Y$ has a λ-scale on Z, equivalently I has a λ-scale on $Z \cap Y \in I^+$; so $F_{Z \cap Y}$ is defined, and by definition F should be a scale in $(\prod A, <_{I \restriction (Z \cap Y)})$, contradicting that F is bounded in $(\prod A, <_{I \restriction Y})$. $\square$

For any infinite cardinal λ, let J_λ be the (possibly improper) ideal of all sets $X \subseteq A$ s.t. $\forall \nu \in \mathrm{pcf}(X)\ \nu < \lambda$. Put another way (because we prefer ideals/filters on A), $X \in J_\lambda$ iff for any ultrafilter D on A, if $X \in D$ then $\mathrm{cf}(\prod A/D) = \mathrm{cf}(\prod A, <_D) < \lambda$. Recall that we defined $\mathrm{pcf}(\varnothing) = \varnothing$, so always $\varnothing \in J_\lambda$. Clearly if λ is a limit cardinal then $J_\lambda = \bigcup_{\nu < \lambda} J_\nu$.

Theorem 5.3 (Fundamental theorem of pcf theory)**.** *If J_λ is proper then it is λ-directed. If $\lambda \in \mathrm{pcf}(A)$ then $J_\lambda \subsetneq J_{\lambda^+}$, and J_{λ^+} is generated by a single set B_λ over J_λ.*

Proof. First observe that if I has a λ-scale or is λ-directed then so is $I \restriction X$ for any $X \in I^+$, or indeed any proper ideal $I' \supseteq I$. By definition, if D is an ultrafitler on A s.t. $D \cap J_\lambda \neq \varnothing$, then $\mathrm{cf}(\prod A/D) = \mathrm{cf}(\prod A, <_D) < \lambda$; we will frequently use the contrapositive, namely if $\mathrm{cf}(\prod A, <_D) \geq \lambda$ then $D \cap J_\lambda = \varnothing$, or equivalently D extends the dual filter of J_λ.

We prove the theorem by induction on λ. If $\lambda \leq \min A$ then $J_\lambda = \{\varnothing\}$, which is trivially $(\min A)$-directed. Also if λ is a limit cardinal and J_ν is a proper ideal for all $\nu < \lambda$, then $J_\lambda = \bigcup_{\nu<\lambda} J_\nu$ is proper because it doesn't contain A, and it is ν-directed for any $\nu < \lambda$, which is the same as λ-directed. If λ is singular then $J_{\lambda^+} = J_\lambda$ and is λ^+-directed.

Now suppose $\lambda \geq \min A > 2^{|A|}$ is regular and J_λ is λ-directed. By the ideal trichotomy, exactly one of the following happens: (i) J_λ has a λ-scale, (ii) J_λ is λ^+-directed, (iii) there exist disjoint J_λ-positive sets X, Y s.t. $A = X \cup Y$, $J_\lambda \restriction X$ has a λ-scale and $J_\lambda \restriction Y$ is λ^+-directed.

In case (i) clearly $\lambda \in \mathrm{pcf}(A)$. In fact $\lambda = \max \mathrm{pcf}(A)$ because if $\mathrm{cf}(\prod A, <_D) \geq \lambda$ then D must extend the dual of J_λ, so D has a λ-scale. Thus $J_{\lambda^+} = A$, and we can let $B_\lambda = A$.

In case (ii), $\lambda \notin \mathrm{pcf}(A)$, since if $\mathrm{cf}(\prod A, <_D) \geq \lambda$ then D has to extend the dual of J_λ, so $(\prod A, <_D)$ is also λ^+-directed and $\mathrm{cf}(\prod A, <_D) \geq \lambda^+$. Therefore $J_{\lambda^+} = J_\lambda$ is λ^+-directed.

In case (iii), we have $X \in J_{\lambda^+}$ because if $D \ni X$ and $\mathrm{cf}(\prod A, <_D) \geq \lambda$ then D must extend the dual of $J_\lambda \restriction X$, so D has a λ-scale. Next we show X generates J_{λ^+} over J_λ. If D doesn't contain X, then either it intersects J_λ, in which case $\mathrm{cf}(\prod A, <_D) < \lambda$, or it extends the dual of $J_\lambda \restriction Y$, and thus $(\prod A, <_D)$ is λ^+-directed. Therefore $\lambda \notin \mathrm{pcf}(A \setminus X)$, and if $E \in J_{\lambda^+}$ then $E \setminus X \in J_\lambda$, which means X generates J_{λ^+} over J_λ, and consequently $J_{\lambda^+} = J_\lambda \restriction Y$ is λ^+-directed. We let $B_\lambda = X$. □

Corollary 5.4. *(i)* $\mathrm{pcf}(A) \leq 2^{|A|}$. *(ii)* $\mathrm{pcf}(A)$ *has a maximal element.*

Proof. (i) The generators B_λ are different for different $\lambda \in \mathrm{pcf}(A)$.

(ii) This is implicit in the proof of the fundamental theorem: if λ is a limit cardinal and J_ν is proper for all $\nu < \lambda$ then J_λ is proper and λ-directed, so there exists $\theta \in \mathrm{pcf}(A)$ s.t. $\theta \geq \lambda$. □

The pcf generator is not unique, but it follows easily from definition that if both B_λ and B'_λ generate J_{λ^+} over J_λ then $B_\lambda = B'_\lambda \mod J_\lambda$; conversely if B_λ is a generator and $B_\lambda = B'_\lambda \mod J_\lambda$ then B'_λ is also a generator. The following characterization of the generator is also useful.

Proposition 5.5. *(i) If* $\lambda \in \mathrm{pcf}(A)$*, then* $X \subseteq A$ *generates* J_{λ^+} *over* J_λ *iff* $\lambda = \max \mathrm{pcf}(X)$ *and* $\lambda \notin \mathrm{pcf}(A \setminus X)$*.*

(ii) If D is an ultrafilter on A, then $\mathrm{cf}(\prod A, <_D) = \lambda$ *iff* $D \ni B_\lambda$ *and D extends the dual filter of* J_λ*. It follows that* $\mathrm{cf}(\prod A, <_D) = \min\{\lambda \in \mathrm{pcf}(A) : B_\lambda \in D\}$*.*

Proof. (i) For the forward direction, we showed in the proof of fundamental theorem that $\lambda = \max \mathrm{pcf}(B_\lambda)$ and $\lambda \notin \mathrm{pcf}(A \setminus B_\lambda)$, so the same is true of any generator X.

Conversely, suppose $\lambda = \max \mathrm{pcf}(X)$ and $\lambda \notin \mathrm{pcf}(A \setminus X)$; the first condition implies $X \in J_{\lambda^+}$. Since $\mathrm{pcf}(B_\lambda \setminus X) \subseteq \mathrm{pcf}(A \setminus X)$, we have $B_\lambda \setminus X \in J_\lambda$ and thus X is a generator.

(ii) If D extends the dual filter of J_λ then it is λ-directed; if it moreover contains B_λ then $\mathrm{cf}(\prod A, <_D)$ cannot exceed λ, and thus is exactly λ.

If $D \cap J_\lambda \neq \varnothing$ then by definition $\mathrm{cf}(\prod A, <_D) < \lambda$. If $B_\lambda \notin D$ then $\mathrm{cf}(\prod A, <_D) \neq \lambda$ since $\lambda \notin \mathrm{pcf}(A \setminus B_\lambda)$. □

Recall that if $|\mathrm{pcf}(A)| < \min A$ then $\overline{A} = \mathrm{pcf}(A)$ is closed under the pcf operation, and thus can be viewed as a topological space; we now know that $2^{|A|} < \min A$ is sufficient for this to be true. Moreover, if $2^{|\mathrm{pcf}(A)|} < \min A = \min \overline{A}$, then the fundamental theorem applies to $\overline{A}$, so there exist generators B_λ for $\lambda \in \mathrm{pcf}(\overline{A}) = \overline{A}$; note that necessarily $\lambda \in B_\lambda$, and also $B_\lambda \subseteq [\min A, \lambda]$. Then $\lambda \notin \mathrm{pcf}(\overline{A} \setminus B_\lambda)$ means λ is not in the closure of $\overline{A} \setminus B_\lambda$, or equivalently B_λ is a neighborhood of λ, namely $B_\lambda \supseteq C \ni \lambda$ for some open set C. Note that C is still a generator.

Conversely, if $C \subseteq \overline{A}$ is an open neighborhood of λ, then $\mathrm{pcf}(B_\lambda \setminus C) \subseteq \mathrm{pcf}(\overline{A} \setminus C) = \overline{A} \setminus C$, so $\lambda \notin B_\lambda \setminus C$ and $B_\lambda \setminus C \in J_\lambda$. It follows that $B_\lambda \cap C = B_\lambda \setminus (B_\lambda \setminus C)$ is a generator, and that the set of generators at λ is a neighborhood basis of λ.

$\overline{A}$ is also Hausdorff, because if $\lambda < \mu$ are in $\overline{A}$ then $B_\mu \setminus B_\lambda$ is still a generator for μ.

Proposition 5.6 (Compactness)**.** *(i) Let B_λ be generators for $\lambda \in \mathrm{pcf}(A)$. For any $X \subseteq A$, there exists a finite set $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathrm{pcf}(X)$ such that $X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$.*

(ii) If $|\mathrm{pcf}(A)| < \min A$, then the space $\overline{A} = \mathrm{pcf}(A)$ is compact.

Proof. (i) Suppose not, then the collection $\{X \setminus B_\lambda : \lambda \in \mathrm{pcf}(X)\}$ has finite intersection property, and hence can be extended to an ultrafilter D on X, which can be viewed as an ultrafilter E on A that is concentrated on X. If $\mathrm{cf}(\prod A/E) = \lambda$ then $\mathrm{cf}(\prod X/D) = \lambda$, so $\lambda \in \mathrm{pcf}(X)$. On the other hand $B_\lambda \in E$ and thus $X \cap B_\lambda \in D$, contradicting the definition of D.

(ii) Suppose $\{X_i : i \in I\}$ is a collection of closed subsets of $\overline{A}$ with finite intersection property. There exists an ultrafilter D on $\overline{A}$ that contains every X_i. If $\lambda = \mathrm{cf}(\prod \overline{A}/D)$ then $\lambda \in \mathrm{pcf}(X_i) = X_i$ for every i, so they have nonempty intersection. $\square$

So far we have been focusing on a fixed A. If $A \subseteq A'$ are sets of regular cardinals satisfying $2^{|A|} < \min A$ and $2^{|A'|} < \min A'$, then there is a simple relationship between their pcf generators. We use superscripts like J^A_λ and B^A_λ for dependence on A.

Proposition 5.7. *Suppose $A \subseteq A'$ are sets of regular cardinals satisfying $2^{|A|} < \min A$ and $2^{|A'|} < \min A'$.*

(i) $J^A_\lambda = J^{A'}_\lambda \cap \mathcal{P}(A)$. If $Y \in J^{A'}_\lambda$ then $Y \cap A \in J^A_\lambda$.

(ii) If $\lambda \in \mathrm{pcf}(A)$ and $B^{A'}_\lambda$ is a generator at λ for A', then $B^{A'}_\lambda \cap A$ is a generator at λ for A.

(iii) For any generator B^A_λ there exists $B^{A'}_\lambda$ s.t. $B^A_\lambda = B^{A'}_\lambda \cap A$.

Proof. (i) By definition J^A_λ is the intersection of $\mathcal{P}(A)$ and the class of all X s.t. $\forall \nu \in \mathrm{pcf}(X)\ \nu \in \lambda$; this shows $J^A_\lambda = J^{A'}_\lambda \cap \mathcal{P}(A)$. If $Y \in J^{A'}_\lambda$ then $Y \cap A \in J^{A'}_\lambda \cap \mathcal{P}(A) = J^A_\lambda$.

(ii) Let $B^A_\lambda = B^{A'}_\lambda \cap A$. Since $\mathrm{pcf}(B^A_\lambda) \subseteq \mathrm{pcf}(B^{A'}_\lambda)$ and $\max \mathrm{pcf}(B^{A'}_\lambda) = \lambda$, we have $B^A_\lambda \in J^A_{\lambda^+}$. Since $B^{A'}_\lambda$ generates $J^{A'}_{\lambda^+}$ and $J^A_\lambda \subseteq J^{A'}_\lambda$, if $X \in J^A_\lambda$ then $X \subseteq Y \cup B^{A'}_\lambda$ for some $Y \in J^{A'}_\lambda$, and thus $X \subseteq (Y \cap A) \cup B^A_\lambda$. By (i) we have $Y \cap A \in J^A_\lambda$, which means B^A_λ is a generator.

(iii) Let B^A_λ and $B^{A'}_\lambda$ be any generators for A and A' respectively. By (ii) we have $B^A_\lambda = B^{A'}_\lambda \cap A \mod J^A_\lambda$, and thus if $C^{A'}_\lambda := (B^{A'}_\lambda \setminus A) \cup B^A_\lambda$ then $C^{A'}_\lambda = B^{A'}_\lambda \mod J^{A'}_\lambda$, so $C^{A'}_\lambda$ is also a generator for A'. $\square$

6 Relation with cardinal arithmetic

Throughout this section let $A = \{\aleph_n : n \in \omega\}$ and assume $\aleph_\omega$ is a strong limit. We will show that $\max \mathrm{pcf}(A) = 2^{\aleph_\omega}$; as outlined before this implies $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$. The method of elementary sub-models is again crucial. Of course the argument generalizes to other A, but we content ourselves with proving this special case. First a simple lemma.

Lemma 6.1. $\mathrm{cf}(\prod A, <) = \max \mathrm{pcf}(A)$*, where* $<$ *means pointwise dominance.*

Proof. For every ultrafilter D on A, let $\lambda = \mathrm{cf}(\prod A, <_D)$ and $F^D = \langle f_\xi^\lambda : \xi < \lambda \rangle$ be a scale in $(\prod A, <_D)$. Let $f \in \prod A$ be arbitrary; we claim that there exists a finite set $\{D_1, \ldots, D_n\}$ of ultrafilters and $\xi_i < \lambda_i = \mathrm{cf}(\prod A, <_{D_i})$ s.t. f is pointwise smaller than $\max_{1 \le i \le n} f_{\xi_i}^{D_i}$. Granted the claim, let $\mu = \max \mathrm{pcf}(A)$; since there are only $2^{2^{|A|}} < \aleph_\omega < \mu$ many ultrafilters, there are only μ many finite combinations of f_ξ^D, so $\mathrm{cf}(\prod A, <) \le \mu$, and the other direction is clear since a $<$-dominating set is also $<_D$-dominating for any D.

If the claim is false, then sets of the form $\{a \in A : f(a) \ge f_\xi^D(a)\}$ have finite intersection property, so can be extended to an ultrafilter E on A. In particular $f_\xi^E \le_E f$ for all $\xi < \lambda = \mathrm{cf}(\prod A, <_E)$, contradicting that F^E is a scale. $\square$

Theorem 6.2. $2^{\aleph_\omega} = \max \mathrm{pcf}(A)$.

Proof. Since we assume $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0} = |[\aleph_\omega]^{\aleph_0}|$, and it suffices to show that $|[\aleph_\omega]^{\aleph_0}| \le \max \mathrm{pcf}(A) := \mu$. By the lemma there exists $F \subseteq \prod A$ s.t. $|F| = \mu$ and F is a dominating set in $(\prod A, <)$. Choose a finite k large enough so that $(2^{\aleph_0})^+ < \aleph_k$. Since $\mathrm{pcf}(A)$ is an interval and $|\mathrm{pcf}(A)| \le 2^{\aleph_0}$, we already know that $\mu = \max \mathrm{pcf}(A) < \aleph_{(2^{\aleph_0})^+} < \aleph_{\aleph_k}$. It is not hard to see that $\aleph_k^{\aleph_0} = \aleph_k$ by the inductive formula for cardinal arithmetic.

Let θ be a large enough regular cardinal and $\prec$ be a well-ordering of H_θ. For each $x \in [\aleph_\omega]^{\aleph_0}$, construct a continuous chain $(M_\alpha^x)_{\alpha < \aleph_k}$ of elementary sub-models of $(H_\theta, \in, \prec)$ such that $F, x \in M_0^x$, $\aleph_k \subseteq M_\alpha^x$, $|M_\alpha^x| = \aleph_k$ and $M_\alpha^x \in M_{\alpha+1}^x$. Define the *characteristic function* χ_α^x on A by $\chi_\alpha^x(\aleph_n) = \sup(M_\alpha^x \cap \aleph_n)$; we have $\chi_\alpha^x(\aleph_n) = \aleph_n$ for $n \le k$ and $\chi_\alpha^x(\aleph_n) < \aleph_n$ for $n > k$ because $|M_\alpha^x| = \aleph_k$, so χ_α^x is almost an element of $\prod A$. Let $M^x = \bigcup_{\alpha < \aleph_k} M_\alpha^x$ and $\chi^x(\aleph_n) = \sup(M^x \cap \aleph_n)$; then $\chi^x = \sup_{\alpha < \aleph_k} \chi_\alpha^x$ pointwise. Clearly if $f \in (\prod A) \cap M_\alpha^x$ then $f < \chi_\alpha^x$ pointwise. Since $M_\alpha^x \in M_{\alpha+1}^x$ we have $\chi_\alpha^x \in M_{\alpha+1}^x$, and thus $\chi_\alpha^x < \chi_{\alpha+1}^x$ pointwise.

We first give an overview of the argument. Consider the composition of maps $x \mapsto M^x \cap \aleph_\omega \mapsto \chi^x$. The first step is $\aleph_k$-to-one because if $M^x \cap \aleph_\omega = M^y \cap \aleph_\omega$ then $y \subseteq M^x$, so there are at most $\aleph_k^{\aleph_0} = \aleph_k$ possibilities for y. The second step (which is well-defined) turns out to be injective, which is the key of the proof. Finally, because of how M^x is constructed, $\langle \chi_\alpha^x : \alpha < \aleph_k \rangle$ is interleaved with a subset of F of size $\aleph_k$, and a relatively simple argument shows μ many subsets suffice, so there are only μ many possibilities for χ^x, and hence $|[\aleph_\omega]^{\aleph_0}| \le \mu$.

So suppose $\sup(M^x \cap \aleph_n) = \sup(M^y \cap \aleph_n)$ for all $n > k$. We shall show that $M^x \cap \aleph_\omega = M^y \cap \aleph_\omega$, by inductively showing $M^x \cap \aleph_n = M^y \cap \aleph_n$. This is true for $n = k$ since $\aleph_k \subseteq M^x$. Now suppose we have proved this for n. Denote $\gamma = \sup(M^x \cap \aleph_{n+1}) = \sup(M^y \cap \aleph_{n+1})$, so γ is an ordinal strictly below $\aleph_{n+1}$ and has cofinality $\aleph_k$; since $M^x \cap \aleph_{n+1}$ and $M^y \cap \aleph_{n+1}$ each contains a club in γ, $C := M^x \cap M^y \cap \aleph_{n+1}$ is also a club in γ. For every $\eta \in C$ s.t. $\eta > \aleph_n$, the $\prec$-least bijection between η and $\aleph_n$ is both in M^x and in M^y by elementarity, and since $M^x \cap \aleph_n = M^y \cap \aleph_n$, translating by the bijection we get $M^x \cap \eta = M^y \cap \eta$; since this is true for every $\eta \in C$ we have $M^x \cap \gamma = M^y \cap \gamma$, in other words $M^x \cap \aleph_{n+1} = M^y \cap \aleph_{n+1}$.

Since $\chi_\alpha^x \in M_{\alpha+1}^x$ and F is cofinal in $\prod A$ under pointwise dominance, by elementarity there exists $f_\alpha^x \in F \cap M_{\alpha+1}^x$ s.t. $\chi_\alpha^x(n) < f_\alpha^x(n)$ for all $n > k$, and we observed previously that $f_\alpha^x < \chi_{\alpha+1}^x$ pointwise. Thus if we let $F^x = \{f_\alpha^x : \alpha < \aleph_k\}$ then F^x is a subset of F of size $\aleph_k$

and $\chi^x(n) = \sup_{\alpha<\aleph_k} \chi^x_\alpha(n) = \sup_{f\in F^x} f(n)$ for all $n > k$; in fact $\chi^x(n) = \sup_{f\in G} f(n)$ for any $G \subseteq F^x$ of size $\aleph_k$. Lastly we show that $\mathrm{cf}([\mu]^{\aleph_k}, \supseteq) = \mu$ (note that it's reverse containment, and the poset is not directed), namely there exists $S \subseteq [\mu]^{\aleph_k}$ with $|S| = \mu$ such that for every $X \subseteq \mu$ with $|X| = \aleph_k$, there exists $Y \in S$ with $Y \subseteq X$. This would finish the proof: replacing μ by F, we have a collection S of $\aleph_k$-subsets of F, with $|S| = \mu$, such that any $X \subseteq F$ with $|X| = \aleph_k$ contains some $Y \in S$. In particular, every F^x contains some $G^x \in S$, and since $|G^x| = \aleph_k$ we have $\chi^x(n) = \sup_{f\in F^x} f(n) = \sup_{f\in G^x} f(n)$ for $n > k$. Consequently there are only μ many χ^x.

We prove $\mathrm{cf}([\mu]^{\aleph_k}, \supseteq) = \mu$ by inductively showing $\mathrm{cf}([\kappa]^{\aleph_k}, \supseteq) = \kappa$ for all cardinals κ between $2^{\aleph_k}$ and μ. This is clear for $2^{\aleph_k}$. If κ is regular then $[\kappa]^{\aleph_k} = \bigcup_{\aleph_k\leq\alpha<\kappa}[\alpha]^{\aleph_k}$; the union is taken over all ordinals α s.t. $\aleph_k \leq \alpha < \kappa$. By induction hypothesis each $[\alpha]^{\aleph_k}$ has a dominating family of size $|\alpha|$, and their union is a dominating family for $[\kappa]^{\aleph_k}$ of size κ. If κ is singular, since $\kappa \leq \mu < \aleph_{\aleph_k}$ we have $\mathrm{cf}(\kappa) < \aleph_k$, and thus if $X \subseteq \kappa$ and $|X| = \aleph_k$ then already $|X \cap \alpha| = \aleph_k$ for some $\alpha < \kappa$, so again the union of dominating families for smaller α gives a dominating family for κ. $\square$

Constructing a chain of elementary sub-models that contain a scale F, and then comparing the characteristic functions to functions in F is a common theme in pcf theory.

Corollary 6.3. *If $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.*

In particular, $2^{\aleph_\omega}$ is not the first weakly inaccessible cardinal. It's not clear how to prove this without using pcf theory in some way.

7 Club guessing and good points

This section collects two technical results: club guessing and the notion of good points. Club guessing is responsible for the mysterious number 4 in Shelah's bound. Good points are used to determine certain cofinalities $\mathrm{cf}(\prod A/I)$ where I is the nonstationary ideal or the ideal of bounded sets. This is another important ingredient in the application to $\aleph_\omega$. Good points also help to relax the condition under which $\mathrm{pcf}(A)$ is an interval, although this is not needed for our application to $\aleph_\omega$.

Club guessing is a very weak version of diamond principle that is provable in ZFC. For $\kappa < \lambda$ regular, denote $E^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$.

Lemma 7.1 (Club guessing). *Let κ, λ be regular uncountable and $\kappa^+ < \lambda$. There is a sequence $\langle c_\alpha : \alpha \in E^\lambda_\kappa\rangle$ such that c_α is a club in α for each $\alpha \in E^\lambda_\kappa$, and for every club $C \subseteq \lambda$, the set $\{\alpha \in E^\lambda_\kappa : C_\alpha \subseteq C\}$ is stationary.*

Proof. For each $\alpha \in E^\lambda_\kappa$, let $c^0_\alpha \subseteq \alpha$ be any club of order type κ. The idea is that if this fails to guess some club C, then we simply intersect each c^0_α with C, and the new sequence guesses C; then we repeat this process; some $c^0_\alpha \cap C$ may no longer be unbounded in α, but most of them remain so.

Inductively, if there exists C s.t. $\{\alpha \in E^\lambda_\kappa \mid c^i_\alpha \subseteq C\}$ is non-stationary, let C_i be this C and $c^{i+1}_\alpha = c^i_\alpha \cap C_i$. When $\gamma < \kappa^+$ is limit let $c^\gamma_\alpha = \bigcap_{i<\gamma} c^i_\alpha = c^0_\alpha \cap \bigcap_{i<\gamma} C_i$. Suppose for contradiction that this construction continues through all $i < \kappa^+$. Since $|c^0_\alpha| = \kappa$, it cannot shrink for κ^+ many times, so c^i_α stabilizes for $i \geq i(\alpha)$ for some $i(\alpha) < \kappa^+$. Since the non-stationary ideal on λ is λ-complete and $\kappa^+ < \lambda$, there is a stationary $S \subseteq E^\lambda_\kappa$ s.t. all $\alpha \in S$ have the same $i(\alpha) = i$. This means for stationarily many α, $c^i_\alpha = c^{i+1}_\alpha$, but the choice of C_i implies that c^i_α has to shrink for almost all $\alpha \in E^\lambda_\kappa$, a contradiction.

So the process stops at some stage $i < \kappa^+$, which means $\langle c^i_\alpha \mid \alpha \in E^\lambda_\kappa \rangle$ satisfies the definition of club-guessing sequence except that c^i_α is only closed, not necessarily a club in α. But $c^i_\alpha = c_\alpha \cap C$ where $C = \bigcap_{j<i} C_j$, and if $\alpha \in C' \cap E^\lambda_\kappa$ (C' is the set of limit points of C, itself a club) then $c_\alpha \cap C$ is a club in α, so the non-club c^i_αs are negligible. $\square$

Suppose A is a set of regular cardinals without maximal element, I is a proper ideal on A that at least contains all the bounded subsets, and $F = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of ordinal functions (not necessarily in $\prod A$).

Definition 7.2. A limit ordinal $\alpha < \lambda$ of uncountable cofinality is *good* (w.r.t. I and F) if $\langle f_\xi : \xi < \alpha \rangle$ has a cofinal subsequence that is pointwise increasing on a large set. More precisely, there exist an unbounded subset $C \subseteq \alpha$ and $Z \in I$ such that for all $\xi < \eta$ in C and all $a \in A \setminus Z$, we have $f_\xi(a) < f_\eta(a)$.

Often but not always I is the ideal of bounded sets, in which case goodness says $f_\xi(a) < f_\eta(a)$ for all $a \geq a_0$ for some a_0. Good points and good scales are related to other "incompacteness principles" such as squares and approachability, and have several variants. For example, if we require the unbounded set C in the above definition to be a club, we get the notion of "very good point".

Lemma 7.3. *Suppose $\alpha \leq \lambda$ is a limit ordinal with $\mathrm{cf}(\alpha) > |I|$. The following are equivalent:*

1. α is good.

2. For every unbounded $C \subseteq \alpha$, there exists an unbounded $D \subseteq C$ and $Z \in I$ such that for all $\xi < \eta$ in D and all $a \in A \setminus Z$, we have $f_\xi(a) < f_\eta(a)$.

3. There exists a sequence $\langle h_i : i < \mathrm{cf}(\alpha) \rangle$ such that:

(i) $\langle h_i : i < \mathrm{cf}(\alpha) \rangle$ is pointwise increasing, i.e., if $i < j < \mathrm{cf}(\alpha)$ then $h_i(a) \leq h_j(a)$ for all $a \in A$.

(ii) $\langle f_\xi : \xi < \alpha \rangle$ is cofinally interleaved with $\langle h_i : i < \mathrm{cf}(\alpha) \rangle$ in $<_I$, i.e., $\forall \xi < \alpha \exists i < \mathrm{cf}(\alpha)\ f_\xi <_I h_i$ and $\forall i < \mathrm{cf}(\alpha) \exists \xi < \alpha\ h_i <_I f_\xi$.

Proof. $2 \to 1$: obvious.

$1 \to 3$: Suppose α is good as witnessed by $Z \in I$ and $C \subseteq \alpha$. We may assume C has order type $\mathrm{cf}(\alpha)$, so enumerate C as $\langle \xi_i : i < \mathrm{cf}(\alpha) \rangle$. Define h_i by $h_i(a) = f_{\xi_i}(a)$ if $a \in A \setminus Z$ and $h_i(a) = i$ if $a \in Z$. Then h_i are pointwise increasing, and $h_i =_I f_{\xi_i}$ so they are cofinally interleaved with $\langle f_\xi : \xi < \alpha \rangle$.

$3 \to 2$: This argument is copied from section 13 of Cummings' survey, and is very similar to Lemma 2.7 in the handbook, where it is called the "sandwich argument". Suppose there exists a sequence $\langle h_i : i < \mathrm{cf}(\alpha) \rangle$ as in 3. For $\xi < \alpha$ let $\iota(\xi)$ be the least $i < \mathrm{cf}(\alpha)$ such that $f_\xi <_I h_i$. Inductively define an increasing sequence $\langle \xi_i : i < \mathrm{cf}(\alpha) \rangle$ of ordinals in C as follows: assuming we have chosen ξ_i, choose ξ_{i+1} so that $h_{\iota(\xi_i)} <_I f_{\xi_{i+1}}$; at limit stage take any upper bound; we may assume C has order type $\mathrm{cf}(\alpha)$ to start with, so that $\langle \xi_i : i < \mathrm{cf}(\alpha)\rangle$ must be unbounded. Since $\mathrm{cf}(\alpha) > |I|$, there exists an unbounded $E \subseteq \mathrm{cf}(\alpha)$ so that for some fixed $Z \in I$, we have

$f_{\xi_{i+1}}(a) < h_{\iota(\xi_{i+1})}(a)$ and $h_{\iota(\xi_i)}(a) < f_{\xi_{i+1}}(a)$ for all $i \in E$ and $a \in A \setminus Z$,

Now Z and $D := \{\xi_{i+1} : i \in E\}$ witnesses that α is good, because for $i < j$ in E we have

$f_{\xi_{i+1}}(a) < h_{\iota(\xi_{i+1})}(a) \le h_{\iota(\xi_j)}(a) < f_{\xi_{j+1}}(a)$ for all $a \in A \setminus Z$,

The middle inequality comes from the fact that $\langle h_i : i < \mathrm{cf}(\alpha)\rangle$ is pointwise increasing, and that $\xi_{i+1} \le \xi_j$, so $\iota(\xi_{i+1}) \le \iota(\xi_j)$.

Hopefully the following picture gives some idea of what is going on. The first row is $\langle f_{\xi_i} : i < \mathrm{cf}(\alpha)\rangle$ (the subscripts are omitted), the second row is $\langle h_{\iota(\xi_i)} : i < \mathrm{cf}(\alpha)\rangle$ and is interleaved with the first row under $<_I$. The boxed f's are those in $\langle f_{\xi_i} : i \in E\rangle$, and their successors are underlined. The first underlined f is smaller than the next h on $A \setminus Z$, and the second underlined f is greater than the previous h on $A \setminus Z$.

$$\begin{array}{ccccccccccccccc} f & & \boxed{f} & & \underline{f} & & f & & f & & \boxed{f} & & \underline{f} & & \cdots \\ & h & & h & & h & & h & & h & & h & & h & \cdots \end{array}$$

□

The following lemma is the key in all our applications of goodness.

Lemma 7.4. *If λ, κ are regular uncountable, $\lambda > \kappa > 2^{|A|}$, $F = \langle f_\xi : \xi < \lambda\rangle$ is such that $\{\alpha \in E^\lambda_\kappa : F$ is good at $\alpha\}$ is stationary, and g is the lub of F, then $\{a \in A : \mathrm{cf}(g(a)) < \kappa\} \in I$.*

Proof. Suppose g is an upper bound such that $X := \{a \in A : \mathrm{cf}(g(a)) < \kappa\} \in I^+$. For every $a \in X$ let S_a be a cofinal subset of $g(a)$ with order type $\mathrm{cf}(g(a))$, in particular $|S_a| < \kappa$. Let $\mathcal{S}$ be the set of all ordinal functions s on A satisfying $s(a) \in S_a$ for all $a \in X$ and $s(a) = g(a)$ elsewhere; we claim there exists $s \in \mathcal{S}$ which is an upper bound of F, and hence g is not an lub.

Suppose the claim fails, then we inductively pick a continuous sequence of ordinals $(\alpha_\xi)_{\xi<\lambda}$ as follows. Since $f_{\alpha_\xi} <_I g$ and S_a is cofinal in $g(a)$, there exists $s_\xi \in \mathcal{S}$ s.t. $f_{\alpha_\xi} <_I s_\xi$. By assumption s_ξ is not an upper bound of F, so choose $\alpha_{\xi+1} > \alpha_\xi$ such that $s_\xi \not\ge_I f_{\alpha_{\xi+1}}$.

Since $E := \{\alpha_\xi : \xi < \lambda\}$ is a club in λ and there are stationarily many good points of cofinality κ, there exists an $\alpha = \alpha_{\xi_0}$ with $\mathrm{cf}(\alpha) = \kappa$ such that α is good and is a limit point of E; it follows that $\mathrm{cf}(\xi_0) = \kappa$. Since $\kappa > 2^{|A|} \ge |I|$, by goodness and the previous lemma, there exists an unbounded subset $C \subseteq \xi_0$ and $Z \in I$ such that $(f_{\alpha_\xi} : \xi \in C)$ is pointwise increasing outside Z; we may assume C has order type κ, but of course it is not a club.

By construction $f_{\alpha_\xi} <_I s_\xi \not\ge_I f_{\alpha_{\xi+1}}$, so $f_{\alpha_\xi}(a) < s_\xi(a) < f_{\alpha_{\xi+1}}(a)$ for all a in some I-positive set that depends on $\xi \in C$. Note that we may assume this positive set is contained in X, since outside X we have $s_\xi(a) = g(a)$ and $f_\xi \le_I g$. For each $\xi \in C$ denote by ξ^+ the next element of C after ξ. Since $\alpha_{\xi+1} \le \alpha_{\xi^+}$, we know that $f_{\alpha_\xi}(a) < s_\xi(a) < f_{\alpha_{\xi^+}}(a)$ on an I-positive subset of X. Since $\kappa > 2^{|A|} \ge |I|$ is regular, there is an unbounded $D \subseteq C$ and a fixed I-positive set $B \subseteq X$ such that

$f_{\alpha_\xi}(a) < s_\xi(a) < f_{\alpha_{\xi^+}}(a)$ for all $\xi \in D$ and $a \in B$

where ξ^+ still denotes successor in C, not D. But since $(f_{\alpha_\xi} : \xi \in C)$ is pointwise increasing outside Z, we have

$s_\xi(a) < f_{\alpha_{\xi^+}}(a) \le f_{\alpha_\eta}(a) < s_\eta(a)$ for all $\xi < \eta$ in D and $a \in B \setminus Z$.

This implies $|\{s_\xi(a) : \xi \in D\}| = \kappa$ for each $a \in B \setminus Z$, contradicting that $|S_a| < \kappa$.

In the following picture, the boxed f's are those in C, so they are pointwise increasing on a large set. Each s is $>_I$ the previous f and $\not\geq_I$ the next f. An s is underlined if its previous f is in D. The two underlined s's are compared using the two "outmost" boxed f's between them, namely the second and the fifth.

$$\begin{array}{cccccccccccccccccccccc}
\boxed{f} & & f & & f & & \boxed{f} & & f & & \boxed{f} & & f & & \boxed{f} & & \boxed{f} & & f & & \cdots & \\
& \underline{s} & & s & & s & & s & & s & & s & & s & & s & & \underline{s} & & s & & \cdots
\end{array}$$
$\square$

Recall we proved that if A is an interval and $(2^{|A})^+ = \min A$ then $\operatorname{pcf}(A)$ is an interval. We can now relax the condition to the following (again, this is not needed in the main result about $\aleph_\omega$).

Proposition 7.5. *If A is an interval of regular cardinals and $2^{|A|} < \min A$, then $\operatorname{pcf}(A)$ is an interval.*

Proof. Let $\mu = \max \operatorname{pcf}(A)$ and $\min A \leq \lambda < \mu$ be regular; we want to show $\lambda \in \operatorname{pcf}(A)$. We may assume $\lambda > \sup A$ since otherwise $\lambda \in A$ and the principal ultrafilter does the job. Let U be an ultrafilter on A s.t. $\operatorname{cf}(\prod A/U) = \mu$; necessarily U is non-principal. By replacing A with the shortest initial segment of A that is in U, we may assume the dual ideal of U contains all bounded subsets of A.

Let $\kappa = \min A > 2^{|A|}$. We will build a $<_U$-increasing sequence $\langle f_\xi : \xi < \lambda\rangle \subseteq \prod A$ that has stationarily many good points of cofinality κ, and thus an eub g satisfying $\operatorname{cf}(g(a)) \geq \min A$ for U-almost all a, and the rest of the proof is the same as before. It turns out in our situation, the naive attempt to make *every* point $\beta < \lambda$ of cofinality κ good actually works (Theroem 2.21 in the handbook uses a weaker hypothesis, but the proof is trickier and involves club guessing). For each $\beta < \lambda$ with cofinality κ, fix a club $C_\beta \subseteq \beta$ of order type κ. Assuming f_ξ has been defined for $\xi < \alpha$, we define f_α in the following way. For every $\beta < \lambda$ with cofinality κ let $h^\beta_\alpha = \sup\{f_\xi : \xi \in C_\beta \cap \alpha\}$; we have $h^\beta_\alpha \in \prod A$ since $|C_\beta \cap \alpha| < \kappa = \min A$. Then choose $f_\alpha \in \prod A$ to be a $<_U$-upper bound of all the h^β_α, which is possible because there are λ many β, the partial order $(\prod A, <_U)$ is μ-directed, and $\lambda < \mu$ by assumption. Then each β is good since $(f_\xi : \xi < \beta)$ is interleaved with $(h^\beta_\alpha : \alpha < \beta)$.

We have built a $<_U$-increasing sequence $\langle f_\xi : \xi < \lambda\rangle \subseteq \prod A$ that has stationarily many good points of cofinality $\kappa = \min A$. Since $\lambda \geq \min A > 2^{|A|}$, the sequence has an eub g, and since $\lambda > 2^{|A|} = |U|$, by the previous lemma we have $\operatorname{cf}(g(a)) \geq \kappa = \min A$ for U-almost all a. Since $\lambda < \mu$, the sequence has a $<_U$ upper bound in $\prod A$, so we may assume $g(a) < a$ everywhere. Thus $\operatorname{cf}(g(a)) \in A$ for U-almost all $a \in A$. Then as before we consider the push-forward ultrafilter, argue that the induced embedding is cofinal, etc. $\square$

Recall that clubs and stationary sets can be defined for any κ with uncountable cofinality (not necessarily regular). They are essentially just clubs or stationary sets on $\operatorname{cf}(\kappa)$, but sometimes it's convenient to deal with κ directly.

In the next two theorems, if X is a set of ordinals, we let $X^+ = \{\aleph_{\alpha+1} : \alpha \in X\}$; conversely, if A is a set of successor cardinals, let $A^- = \{\alpha : \aleph_{\alpha+1} \in A\}$.

Theorem 7.6. *Suppose $\aleph_\eta$ is a singular cardinal with $\operatorname{cf}(\eta) = \tau > \omega$, $2^\tau < \aleph_\eta$, $S \subseteq \eta$ is a stationary set of order type τ, $A = S^+$ and I is the ideal on A defined by $I = \{X^+ : X \subseteq S$ is non-stationary$\}$.*

Then $\mathrm{tcf}(\prod A/I) = \aleph_{\eta+1}$.

Proof. Let $\lambda = \aleph_{\eta+1}$. Clearly $(\prod A, <_I)$ is λ-directed. Assume for contradiction that it does not have a λ-scale; then by ideal trichotomy there exists $T \subseteq S$ stationary such that if $J = I \restriction T^+$ then $(\prod A, <_J)$ is λ^+-directed. We construct a $<_J$-increasing sequence $F = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod A$ for which all points are good. Every limit ordinal $\beta < \lambda$ has cofinality below λ, and hence below $\aleph_\eta$, so fix a club $C_\beta \subseteq \beta$ of order type $\mathrm{cf}(\beta) < \aleph_\eta$. Assume f_ξ has been defined for $\xi < \alpha$. For every $\beta < \lambda$ let $h^\beta_\alpha = \sup\{f_\xi : \xi \in C_\beta \cap \alpha\}$; since $|C_\beta| < \aleph_\eta$, we have $\forall a \geq a_0\ h^\beta_\alpha(a) < a$ for some $a_0 \in A$ (note that a_0 depends on β but not on α), and thus $h^\beta_\alpha \in \prod A$ after we redefine its value on a bounded set. Since $<_J$ is λ^+-directed, $\{h^\beta_\alpha : \beta < \lambda\} \cup \{f_\xi : \xi < \alpha\}$ has an upper bound, and we let f_α be $<_J$-greater than this upper bound. The resulting F is good at every β, because $\{f_\xi : \xi < \beta\}$ is $<_J$-interleaved with $\{h^\beta_\alpha : \alpha < \beta\}$, and the latter is pointwise increasing.

Since $2^{|A|} = 2^\tau < \aleph_\eta < \lambda$, F has an eub g; on the other hand, $(\prod A, <_J)$ is λ^+-directed, which means F has an $<_J$-upper bound in $\prod A$, so we may assume $g \in \prod A$, namely $g(a) < a$. Consequently, $\alpha \mapsto \mathrm{cf}(g(\aleph_{\alpha+1}))$ is (almost) a regressive function on S, because $g(\aleph_{\alpha+1}) < \aleph_{\alpha+1}$ implies $g(\aleph_{\alpha+1}) < \aleph_\alpha$, unless $\aleph_\alpha$ is regular, which is rare: it is not hard to show that $\{\alpha \in \eta : \aleph_\alpha \text{ is singular}\}$ is a club in η. By Fodor's lemma, $\{\alpha \in T : \mathrm{cf}(g(\aleph_{\alpha+1})) = \kappa\}$ is stationary for some $\kappa < \aleph_\eta$, but this contradicts goodness and Lemma 7.4, which say $\{a \in A : \mathrm{cf}(g(a)) < \kappa\} \in J$ for every $2^\tau < \kappa < \lambda$. □

Theorem 7.7. *Suppose $\aleph_\eta$ is a singular cardinal with $\mathrm{cf}(\eta) = \tau > \omega$ and $2^\tau < \aleph_\eta$. There exists a club $D \subseteq \eta$ of order type τ so that $\mathrm{tcf}(\prod D^+/J) = \max \mathrm{pcf}(D^+) = \aleph_{\eta+1}$, where J is the ideal on D^+ of bounded sets.*

Proof. First let $C \subseteq \eta$ be any club of order type τ, $A = C^+$ and I be the ideal on A corresponding to the non-stationary ideal. By the previous theorem $\mathrm{tcf}(\prod A/I) = \aleph_{\eta+1} =: \lambda$. We may choose the first element of C large enough so that $2^{|A|} = 2^\tau < \min A$, so there exist pcf generators for A. In particular there is a generator B_λ. The previous theorem also implies $\lambda \in \mathrm{pcf}(S^+)$ for any stationary $S \subseteq \eta$. Since $\lambda \notin \mathrm{pcf}(A \setminus B_\lambda)$, $(A \setminus B_\lambda)^-$ cannot be stationary, which means $(B_\lambda)^-$ contains a club $D \subseteq \eta$. Then $\mathrm{tcf}(\prod D^+/I') = \max \mathrm{pcf}(D^+) = \lambda$, where $I' = I \cap \mathcal{P}(D^+)$.

To show that $\mathrm{tcf}(\prod D^+/J) = \max \mathrm{pcf}(D^+)$, note that B_ν is bounded in A for every $\nu < \lambda$, so the ideal J_{bd} on A of bounded sets extends the ideal generated by $\{B_\nu : \nu < \lambda\}$, which is J_λ; since J_λ is λ-directed, so is J_{bd}. The same holds for the bounded ideal J on D^+. Since $\max \mathrm{pcf}(D^+) = \lambda$, J must have a λ-scale by ideal trichotomy. □

8 Transitive generators

This section presents one of the most technical results, the existence of *transitive generators* B_λ. Transitivity means if $\mu \in B_\lambda$ then $B_\mu \subseteq B_\lambda$, in other words the relation $\mu \in B_\lambda$ is transitive. Note that this is trivial for $A = \{\aleph_n : n < \omega\}$, but we want to apply the result to $\overline{A} := \mathrm{pcf}(A)$, obtaining the corollary that the pcf space $\overline{A}$ has a tightness property. Recall that a generator B_λ for $\lambda \in \mathrm{pcf}(\overline{A}) = \overline{A}$ is in particular a neighborhood of λ in $\overline{A}$, so if the generators are transitive then they are actually open.

A naive attempt to obtain transitive operators is to take transitive closure, namely let $\overline{B}_\lambda$ consists of all ν for which there is a sequence $(\nu_0, \dots, \nu_n)$ such that $\nu_0 = \nu$, $\nu_n = \lambda$, and $\nu_k \in B_{\nu_{k+1}}$. In general this can mess things up and $\overline{B}_\lambda$ may not be a generator, but it turns out if we first shrink B_λ in a certain way and then take transitive closure then it works. The shrinking process involves elementary sub-models.

Theorem 8.1. *Suppose A is a set of regular cardinals such that $(2^{|A|})^+ < \min A$, then there exist transitive generators B_λ for $\lambda \in \mathrm{pcf}(A)$.*

Proof. First let $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be any generators. For each $\lambda \in \mathrm{pcf}(A)$ let $F^\lambda = \langle f^\lambda_\alpha : \alpha < \lambda\rangle$ be cofinal in $(\prod A, <_{J_\lambda \restriction B_\lambda})$; we moreover require that f^λ_β is the eub of $\langle f^\lambda_\alpha : \alpha < \beta\rangle$ whenever $\mathrm{cf}(\beta) > 2^{|A|}$.

Let $\kappa = (2^{|A|})^+$ and $(M_i)_{i<\kappa}$ be a continuous chain of elementary sub-models such that $|M_i| = \kappa$, $M_i \in M_{i+1}$, $\kappa \subseteq M_0$ and M_0 contains A, $\mathcal{P}(A)$, $(F^\lambda : \lambda \in \mathrm{pcf}(A))$ and all functions from some subset of A to $A^{<\omega}$ (the purpose of this will become clear later). Let $M = \bigcup_{i<\kappa} M_i$. Note that if $X \in M$ and $|X| \le \kappa$ then $X \subseteq M$; in particular $\mathrm{pcf}(A) \subseteq M$ and $F^\lambda \in M$ for every λ, although M only contains a small portion of $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$. Define the characteristic functions $\chi_i(\lambda) = \sup(M_i \cap \lambda)$ and $\chi = \sup(M \cap \lambda) = \sup_{i<\kappa} \chi_i$ on A; note that they belong to $\prod A$ since $\kappa < \min A$.

Since $M_i \in M_{i+1}$ and χ_i is definable from M_i, by elementarity $\chi_i \in M_{i+1}$; since F^λ is cofinal, again by elementarity there exists $\alpha \in M_{i+1}$, $\alpha < \lambda$ s.t. $\chi_i <_{J_\lambda \restriction B_\lambda} f^\lambda_\alpha$. Thus for each λ, $\langle \chi_i : i < \kappa\rangle$ and $\langle f^\lambda_\alpha : \alpha < \chi(\lambda)\rangle$ are cofinally interleaved in $<_{J_\lambda \restriction B_\lambda}$. By assumption $f^\lambda_{\chi(\lambda)}$ is the eub for the latter. On the other hand χ is the eub for $\langle \chi_i : i < \kappa\rangle$ in $<_{J_\lambda \restriction B_\lambda}$ since it is already the eub under pointwise dominance $<$. In detail, if $h <_{J_\lambda \restriction B_\lambda} \chi$ then $h(a) < \chi(a)$ on a $J_\lambda \restriction B_\lambda$-large set X; for each $a \in X$ there exists an $i < \kappa$ s.t. $h(a) < \chi_i(a)$, and since $|X| \le |A| < \kappa$ we know that $h < \chi_i$ pointwise for some i, and thus $h <_{J_\lambda \restriction B_\lambda} \chi_i$. Consequently, for every $\lambda \in \mathrm{pcf}(A)$ we have

$f^\lambda_{\chi(\lambda)} = \chi \mod J_\lambda \restriction B_\lambda$, and thus $B^*_\lambda := \{\nu \in B_\lambda : f^\lambda_{\chi(\lambda)}(\nu) = \chi(\nu)\}$ is a generator.

Let $\langle \overline{B}_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be the transitive closure, namely $\overline{B}_\lambda$ is the set of all ν for which there exists a sequence $(\nu_0, \dots, \nu_n)$ such that $\nu_0 = \nu$, $\nu_n = \lambda$, and $\nu_k \in B^*_{\nu_{k+1}}$ for $0 \le k \le n-1$; the case $n = 0$ implies $\lambda \in \overline{B}_\lambda$ and $n = 1$ implies $B^*_\lambda \subseteq \overline{B}_\lambda$.

To show that each $\overline{B}_\lambda$ is a generator, it suffices to show that $\overline{B}_\lambda \in J_{\lambda^+}$. We do this by finding a function g such that $g <_{J_{\lambda^+}} \chi$ while $g(\nu) = \chi(\nu)$ for all $\nu \in \overline{B}_\lambda$. As an illustration, let us first try to find such a g that works for the case $n = 2$, namely $g(\nu_0) = \chi(\nu_0)$ for all $\nu_0 \in \overline{B}^{(2)}_\lambda$, where $\overline{B}^{(2)}_\lambda = \bigcup_{\nu_1 \in B^*_\lambda} B^*_{\nu_1}$. Fix for every $\nu_0 \in \overline{B}^{(2)}_\lambda$ some ν_1 such that $\nu_0 \in B^*_{\nu_1}$ and $\nu_1 \in B^*_\lambda$. By definition of $B^*_{\nu_1}$ and B^*_λ we have

$f^\lambda_{\chi(\lambda)}(\nu_1) = \chi(\nu_1)$ and $f^{\nu_1}_{\chi(\nu_1)}(\nu_0) = \chi(\nu_0)$, so $f^{\nu_1}_{f^\lambda_{\chi(\lambda)}(\nu_1)}(\nu_0) = \chi(\nu_0)$.

Define $g_\alpha(\nu_0) = f^{\nu_1}_{f^\lambda_\alpha(\nu_1)}(\nu_0)$ if $\nu_0 \in \overline{B}^{(2)}_\lambda$ and $g_\alpha = 0$ elsewhere on A. Then we have shown that $g_{\chi(\lambda)} = \chi$ on $\overline{B}^{(2)}_\lambda$. Even though $g_{\chi(\lambda)}$ is not in M, it can be argued that the sequence $(g_\alpha : \alpha < \lambda)$ is. Since J_{λ^+} is λ^+-directed, we can pick an upper bound $h \in M$ of $(g_\alpha : \alpha < \lambda)$, so in particular $g_{\chi(\lambda)} <_{J_{\lambda^+}} h < \chi$. Thus $\overline{B}^{(2)}_\lambda \in J_{\lambda^+}$.

It should seem at least plausible that this argument generalizes to arbitrary n, albeit at the cost of a notational disaster. We now proceed to the actual proof. Fix a function φ such that for each $\nu \in \overline{B}_\lambda$, $\varphi(\nu)$ is a sequence $(\nu_0, \dots, \nu_n)$ such that $\nu_0 = \nu$, $\nu_n = \lambda$, and $\nu_k \in B^*_{\nu_{k+1}}$ for $0 \le k \le n-1$. For each $\alpha < \lambda$ define g_α on A as follows; if $\nu \notin \overline{B}_\lambda$ then $g_\alpha(\nu) = 0$. If $\nu \in \overline{B}_\lambda$ then let $\varphi(\nu) = (\nu_0, \dots, \nu_n)$ and inductively define a sequence $(\beta_n, \dots, \beta_0)$ by $\beta_n = \alpha$ and $\beta_k = f^{\nu_{k+1}}_{\beta_{k+1}}(\nu_k)$ for $0 \le k \le n-1$; the expression is meaningful because inductively we have $\beta_k < \nu_k$. Define $g_\alpha(\nu) = \beta_0$. It can be checked that when $n = 2$ this agrees with the explicit formula above, and it should be clear now why we don't explicitly write down the case $n = 3$.

Recall that M_0 contains all functions from some subset of A to $A^{<\omega}$, in particular $\varphi \in M$, so the sequence $(g_\alpha : \alpha < \lambda)$ is also in M, because it is definable from φ and F^λ. Since J_{λ^+} is λ^+-directed, there is in M some $h \in \prod A$ such that $g_\alpha <_{J_{\lambda^+}} h$ for all α, in particular $g_{\chi(\lambda)} <_{J_{\lambda^+}} h < \chi$. We finish the proof by showing $g_{\chi(\lambda)} = \chi$ on $\overline{B}_\lambda$. If $\nu \in \overline{B}_\lambda$ and $\varphi(\nu) = (\nu_0, \dots, \nu_n)$, then $\beta_n = \chi(\lambda) = \chi(\nu_n)$ and inductively

$$\beta_k = f^{\nu_{k+1}}_{\beta_{k+1}}(\nu_k) = f^{\nu_{k+1}}_{\chi(\nu_{k+1})}(\nu_k) = \chi(\nu_k),$$

so $g_{\chi(\lambda)}(\nu) = \beta_0 = \chi(\nu_0) = \chi(\nu)$. □

Theorem 8.2 (Localization). *Suppose A is a set of regular cardinals, $2^{|\mathrm{pcf}(A)|} < \min A$, $X \subseteq \mathrm{pcf}(A)$ and $\lambda \in \mathrm{pcf}(X)$. Then there exists $W \subseteq X$ such that $|W| \le |A|$ and $\lambda \in \mathrm{pcf}(W)$.*

Proof. As usual we may assume $(2^{|\mathrm{pcf}(A)|})^+ < \min A$ by possibly removing the first point of A. Let $\overline{A} = \mathrm{pcf}(A)$; then $(2^{|\overline{A}|})^+ < \min \overline{A}$, so there exist transitive generators B_ν for $\nu \in \mathrm{pcf}(\overline{A}) = \overline{A}$; in particular $\max \mathrm{pcf}(B_\lambda) = \lambda$. Let $Y = X \cap B_\lambda$; since $\mathrm{pcf}(X) = \mathrm{pcf}(X \setminus B_\lambda) \cup \mathrm{pcf}(Y)$ and $\lambda \notin \mathrm{pcf}(X \setminus B_\lambda)$, we have $\lambda \in \mathrm{pcf}(Y)$, and thus $\lambda = \max \mathrm{pcf}(Y)$.

Recall that $B_\nu \cap A$ is a generator for $\nu \in \mathrm{pcf}(A)$. Let $E = \bigcup_{\nu \in Y} B_\nu \cap A$; in particular $\nu \in \mathrm{pcf}(B_\lambda \cap A) \subseteq \mathrm{pcf}(E)$ for every $\nu \in Y$, so $Y \subseteq \mathrm{pcf}(E)$ and $\lambda \in \mathrm{pcf}(Y) \subseteq \mathrm{pcf}(\mathrm{pcf}(E)) = \mathrm{pcf}(E)$. Since $E \subseteq A$, there exists $W \subseteq Y$ with $|W| \le |A|$ such that $E = \bigcup_{\nu \in W} B_\nu \cap A$. We shall show that $\lambda \in \mathrm{pcf}(W)$. Suppose not; since $\overline{A}$ is a compact space, $\mathrm{pcf}(W)$ is closed and for each $\nu \in \mathrm{pcf}(W)$, B_ν is an open neighborhood of ν, there exist finitely many $\nu_1, \dots, \nu_n \in \mathrm{pcf}(W)$ such that $W \subseteq \mathrm{pcf}(W) \subseteq B_{\nu_1} \cup \dots \cup B_{\nu_n}$. So

$$E \subseteq \textstyle\bigcup_{\nu \in W} B_\nu \subseteq \bigcup_{i=1}^n \bigcup_{\nu \in B_{\nu_i}} B_\nu \subseteq \bigcup_{i=1}^n B_{\nu_i},$$

the last step due to transitivity. Since $W \subseteq Y$, $\lambda = \max \mathrm{pcf}(Y)$ and $\lambda \notin \mathrm{pcf}(W)$, all the ν_i are less than λ, so from $\mathrm{pcf}(E) \subseteq \bigcup_{i=1}^n \mathrm{pcf}(B_{\nu_i})$ we get $\lambda \notin \mathrm{pcf}(E)$, a contradiction. □

In topological terms, if λ is in the closure of X then it is in the closure of a relatively small subset of X, namely of size at most $|A|$. This is called $|A|$-tightness. Note that $\overline{A}$ has a dense set of size $|A|$, namely A, but this does not imply $|A|$-tightness in general. A topological space X equipped with a well-ordering $<$ is called right separated if every initial segment $\{x \in X : x < y\}$ is open. The above proof can be rephrased in topological terms as follows: if a compact right separated space X has a dense set of size κ, as well as a neighborhood U_x for each $x \in X$ satisfying $x \in U_y \to U_x \subseteq U_y$ and $U_x \subseteq \{z \in X : z \le x\}$, then it is κ-tight.

9 The number 4

Theorem 9.1. *If $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Proof. Let $A = \{\aleph_n : n \in \omega\}$. It suffices to show that $|\mathrm{pcf}(A)| < \omega_4$. Since we already know that $\overline{A} = \mathrm{pcf}(A)$ is an interval and $\max \overline{A} < \aleph_{(2^{\aleph_0})^+} < \aleph_{\aleph_\omega}$, $\overline{A}$ is the set of all successor cardinals between $\aleph_0$ and $\max \overline{A} = \aleph_{\vartheta+1}$ for some $\vartheta < \aleph_\omega$. Define a function $F : \mathcal{P}(\vartheta+1) \to \vartheta+1$ as follows; if $X \subseteq \vartheta+1$ then $\{\aleph_{\alpha+1} : \alpha \in X\} \subseteq \overline{A}$, and $\max \mathrm{pcf}\{\aleph_{\alpha+1} : \alpha \in X\}$ is some successor cardinal $\aleph_{\beta+1} \leq \aleph_{\vartheta+1}$; define $F(X)$ to be this β. Then the function F has the following properties:

(i) If $X \subseteq Y$ then $F(X) \leq F(Y)$.

(ii) If $\eta \leq \vartheta$ has uncountable cofinality, then there is a club $C \subseteq \eta$ such that $F(C) = \eta$. This follows from the last theorem in section 7; note that $\tau := \mathrm{cf}(\eta) < \aleph_\omega$, so $2^\tau < \aleph_\omega < \aleph_\eta$.

(iii) If X has order type ω_1, then there exists $\gamma < \sup X$ such that $F(X \cap \gamma) = F(X)$. This follows from localization theorem. Also note that $F(X) \geq \sup X$ always holds.

Suppose for contradiction that $\vartheta \geq \omega_4$. Let $\langle C_\alpha : \alpha \in E^{\omega_3}_{\omega_1} \rangle$ be a club guessing sequence. Let $(M_\alpha)_{\alpha<\omega_3}$ be a continuous chain of elementary sub-models such that $|M_\alpha| = \omega_3$, $M_\alpha \cap \omega_4 \in \omega_4$, $(M_\beta : \beta \leq \alpha) \in M_{\alpha+1}$, M_0 contains the club guessing sequence and the function F; note that each C_α is in M_0 but $\mathrm{pcf}(A) \nsubseteq M_0$. Let $\eta_\alpha = M_\alpha \cap \omega_4$, so $\eta = \sup_{\alpha<\omega_3} \eta_\alpha$ satisfies $\omega_3 < \eta < \omega_4 \leq \vartheta$, and has cofinality ω_3; by (ii) there is a club $C \subseteq \eta$ such that $F(C) = \eta$. The intersection of C with $\{\eta_\alpha : \alpha < \omega_3\}$ is also a club; by club guessing there exists $\gamma \in E^{\omega_3}_{\omega_1}$ such that $Y := \{\eta_\alpha : \alpha \in C_\gamma\} \subseteq C$. By (iii) there exists $\beta < \gamma$ such that if $X = \{\eta_\alpha : \alpha \in C_\gamma \cap \beta\}$ then $F(X) = F(Y) \geq \sup Y = \eta_\gamma$.

We claim that $X \in M_\gamma$. First we have $C_\gamma \in M_0 \subseteq M_\gamma$; although the function $\alpha \mapsto \eta_\alpha$ is not in M_γ, its restriction to β is, because $(M_\alpha : \alpha < \beta) \in M_{\beta+1} \subseteq M_\gamma$. Hence $F(X) \in M_\gamma$; on the other hand $F(X) = F(Y) \leq F(C) = \eta < \omega_4$, which means $F(X) < \sup(M_\gamma \cap \omega_4) = \eta_\gamma$, a contradiction. □